Chapter 10

Complex Vectors and Matrices

10.1 Complex Numbers

A complete presentation of linear algebra must include complex numbers. Even when the matrix is real, ***the eigenvalues and eigenvectors are often complex***. Example: A 2 by 2 rotation matrix has no real eigenvectors. Every vector in the plane turns by θ—its direction changes. But the rotation matrix has complex eigenvectors $(1, i)$ and $(1, -i)$.

Notice that those eigenvectors are connected by changing i to $-i$. For a real matrix, the eigenvectors come in "conjugate pairs." The eigenvalues of rotation by θ are also conjugate complex numbers $e^{i\theta}$ and $e^{-i\theta}$. We must move from $\mathbf{R}^n$ to $\mathbf{C}^n$.

The second reason for allowing complex numbers goes beyond λ and x to the matrix A. ***The matrix itself may be complex.*** We will devote a whole section to the most important example—*the Fourier matrix.* Engineering and science and music and economics all use Fourier series. In reality the series is finite, not infinite. Computing the coefficients in $c_1e^{ix} + c_2e^{i2x} + \cdots + c_ne^{inx}$ is a linear algebra problem.

This section gives the main facts about complex numbers. It is a review for some students and a reference for everyone. Everything comes from $i^2 = -1$. The Fast Fourier Transform applies the amazing formula $e^{2\pi i} = 1$. Add angles when $e^{i\theta}$ multiplies $e^{i\theta}$:

The square of $e^{2\pi i/4} = i$ is $e^{4\pi i/4} = -1$. The fourth power of $e^{2\pi i/4}$ is $e^{2\pi i} = 1$.

Adding and Multiplying Complex Numbers

Start with the imaginary number i. Everybody knows that $x^2 = -1$ has no real solution. When you square a real number, the answer is never negative. So the world has agreed on a solution called i. (Except that electrical engineers call it j.) Imaginary numbers follow the normal rules of addition and multiplication, with one difference. ***Replace i^2 by -1.***

A complex number (say $3 + 2i$) ***is the sum of a real number*** (3) ***and a pure imaginary number*** ($2i$). Addition keeps the real and imaginary parts separate. Multiplication uses $i^2 = -1$:

$$\textbf{Add:} \quad (3+2i) + (3+2i) = 6 + 4i$$
$$\textbf{Multiply:} \quad (3+2i)(1-i) = 3 + 2i - 3i - 2i^2 = 5 - i.$$

If I add $3 + i$ to $1 - i$, the answer is 4. The real numbers $3 + 1$ stay separate from the imaginary numbers $i - i$. We are adding the vectors $(3, 1)$ and $(1, -1)$.

The number $(1 + i)^2$ is $1 + i$ times $1 + i$. The rules give the surprising answer $2i$:

$$(1 + i)(1 + i) = 1 + i + i + i^2 = 2i.$$

In the complex plane, $1+i$ is at an angle of 45°. It is like the vector $(1, 1)$. When we square $1 + i$ to get $2i$, the angle doubles to 90°. If we square again, the answer is $(2i)^2 = -4$. The 90° angle doubled to 180°, the direction of a negative real number.

A real number is just a complex number $z = a + bi$, with zero imaginary part: $b = 0$. A pure imaginary number has $a = 0$:

The ***real part*** is $a = \text{Re}\,(a + bi)$. The ***imaginary part*** is $b = \text{Im}\,(a + bi)$.

The Complex Plane

Complex numbers correspond to points in a plane. Real numbers go along the x axis. Pure imaginary numbers are on the y axis. ***The complex number $3 + 2i$ is at the point with coordinates*** $(\mathbf{3}, \mathbf{2})$. The number zero, which is $0 + 0i$, is at the origin.

Adding and subtracting complex numbers is like adding and subtracting vectors in the plane. The real component stays separate from the imaginary component. The vectors go head-to-tail as usual. The complex plane $\mathbf{C}^1$ is like the ordinary two-dimensional plane $\mathbf{R}^2$, except that we multiply complex numbers and we didn't multiply vectors.

Now comes an important idea. ***The complex conjugate of $3 + 2i$ is $3 - 2i$.*** The complex conjugate of $z = 1 - i$ is $\overline{z} = 1 + i$. In general the conjugate of $z = a + bi$ is $\overline{z} = a - bi$. **(Some writers use a "*bar*" on the number and others use a "*star*":** $\overline{z} = z^*$.) The imaginary parts of z and "z bar" have opposite signs. In the complex plane, $\overline{z}$ is the image of z on the other side of the real axis.

Two useful facts. ***When we multiply conjugates $\overline{z}_1$ and $\overline{z}_2$, we get the conjugate of $z_1 z_2$.*** When we add $\overline{z}_1$ and $\overline{z}_2$, we get the conjugate of $z_1 + z_2$:

$$\overline{z}_1 + \overline{z}_2 = (3 - 2i) + (1 + i) = 4 - i. \text{ This is the conjugate of } z_1 + z_2 = 4 + i.$$
$$\overline{z}_1 \times \overline{z}_2 = (3 - 2i) \times (1 + i) = 5 + i. \text{ This is the conjugate of } z_1 \times z_2 = 5 - i.$$

Adding and multiplying is exactly what linear algebra needs. By taking conjugates of $Ax = \lambda x$, when A is real, we have another eigenvalue $\overline{\lambda}$ and its eigenvector $\overline{x}$:

$$\textit{If } Ax = \lambda x \textit{ and } A \textit{ is real then } A\overline{x} = \overline{\lambda}\,\overline{x}. \tag{1}$$

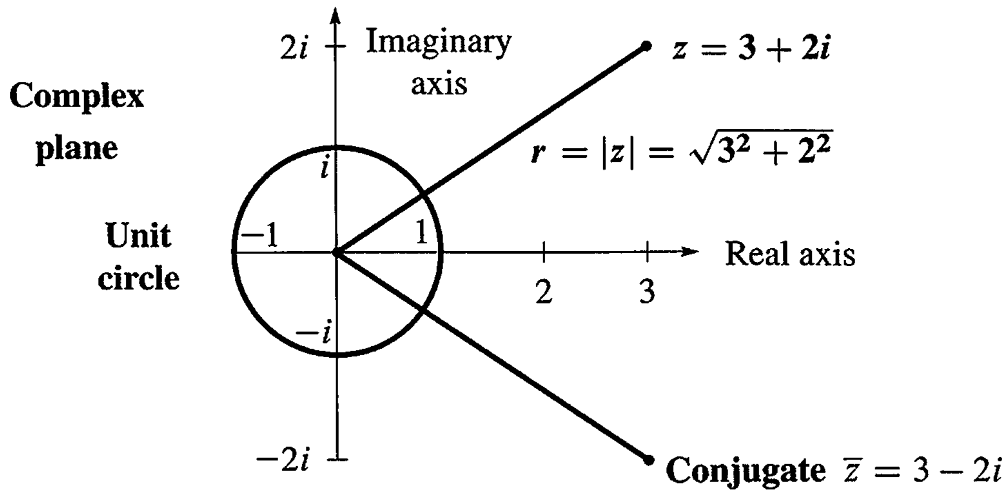


Figure 10.1: The number $z = a + bi$ corresponds to the point (a, b) and the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

Something special happens when $z = 3 + 2i$ combines with its own complex conjugate $\bar{z} = 3 - 2i$. The result from adding $z + \bar{z}$ or multiplying $z\bar{z}$ is always real:

$$\begin{aligned} z + \bar{z} &= \textbf{real} \qquad (3+2i) + (3-2i) = 6 \quad \text{(real)} \\ z\bar{z} &= \textbf{real} \qquad (3+2i) \times (3-2i) = 9 + 6i - 6i - 4i^2 = 13 \quad \text{(real)}. \end{aligned}$$

The sum of $z = a + bi$ and its conjugate $\bar{z} = a - bi$ is the real number $2a$. The product of z times $\bar{z}$ is the real number $a^2 + b^2$:

Multiply z times $\bar{z}$

$$(a + bi)(a - bi) = a^2 + b^2. \tag{2}$$

The next step with complex numbers is $1/z$. How to divide by $a + ib$? The best idea is to multiply by $\bar{z}/\bar{z}$. That produces $z\bar{z}$ in the denominator, which is $a^2 + b^2$:

$$\frac{1}{a+ib} = \frac{1}{a+ib}\frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} \qquad \frac{1}{3+2i} = \frac{1}{3+2i}\frac{3-2i}{3-2i} = \frac{3-2i}{13}.$$

In case $a^2 + b^2 = 1$, this says that $(a + ib)^{-1}$ is $a - ib$. ***On the unit circle,* $1/z$ equals $\bar{z}$.** Later we will say: $1/e^{i\theta}$ is $e^{-i\theta}$ (the conjugate). A better way to multiply and divide is to use the polar form with distance r and angle θ.

The Polar Form $re^{i\theta}$

The square root of $a^2 + b^2$ is $|z|$. This is the ***absolute value*** (or ***modulus***) of the number $z = a + ib$. The square root $|z|$ is also written r, because it is the distance from 0 to z. ***The real number r in the polar form gives the size of the complex number z:***

The absolute value of $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$. This is called r.

The absolute value of $z = 3 + 2i$ is $|z| = \sqrt{3^2 + 2^2}$. This is $r = \sqrt{13}$.

The other part of the polar form is the angle θ. The angle for $z = 5$ is $\theta = 0$ (because this z is real and positive). The angle for $z = 3i$ is $\pi/2$ radians. The angle for a negative $z = -9$ is π radians. ***The angle doubles when the number is squared.*** The polar form is excellent for multiplying complex numbers (not good for addition).

When the distance is r and the angle is θ, trigonometry gives the other two sides of the triangle. The real part (along the bottom) is $a = r\cos\theta$. The imaginary part (up or down) is $b = r\sin\theta$. Put those together, and the rectangular form becomes the polar form:

The number $\boldsymbol{z = a + ib}$ ***is also*** $\boldsymbol{z = r\cos\theta + ir\sin\theta}$. ***This is*** $\boldsymbol{re^{i\theta}}$

Note: $\boldsymbol{\cos\theta + i\sin\theta}$ ***has absolute value*** $\boldsymbol{r = 1}$ ***because*** $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$. Thus $\cos\theta + i\sin\theta$ lies on the circle of radius 1—*the unit circle.*

Example 1 Find r and θ for $z = 1 + i$ and also for the conjugate $\bar{z} = 1 - i$.

Solution The absolute value is the same for z and $\bar{z}$. For $z = 1+i$ it is $r = \sqrt{1+1} = \sqrt{2}$:

$$|z|^2 = 1^2 + 1^2 = 2 \qquad \text{and also} \qquad |\bar{z}|^2 = 1^2 + (-1)^2 = 2.$$

The distance from the center is $\sqrt{2}$. What about the angle? The number $1 + i$ is at the point $(1, 1)$ in the complex plane. The angle to that point is $\pi/4$ radians or 45°. The cosine is $1/\sqrt{2}$ and the sine is $1/\sqrt{2}$. Combining r and θ brings back $z = 1 + i$:

$$r\cos\theta + ir\sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1 + i.$$

The angle to the conjugate $1 - i$ can be positive or negative. We can go to $7\pi/4$ radians which is 315°. Or we can go *backwards through a negative angle*, to $-\pi/4$ radians or $-45°$. ***If z is at angle*** $\boldsymbol{\theta}$***, its conjugate*** $\boldsymbol{\bar{z}}$ ***is at*** $\boldsymbol{2\pi - \theta}$ ***and also at*** $\boldsymbol{-\theta}$.

We can freely add 2π or 4π or -2π to any angle! Those go full circles so the final point is the same. This explains why there are infinitely many choices of θ. Often we select the angle between zero and 2π radians. But $-\theta$ is very useful for the conjugate $\bar{z}$.

Powers and Products: Polar Form

Computing $(1 + i)^2$ and $(1 + i)^8$ is quickest in polar form. That form has $r = \sqrt{2}$ and $\theta = \pi/4$ (or 45°). If we square the absolute value to get $r^2 = 2$, and double the angle to get $2\theta = \pi/2$ (or 90°), we have $(1 + i)^2$. For the eighth power we need r^8 and 8θ:

$$\boldsymbol{(1+i)^8} \qquad r^8 = 2\cdot 2\cdot 2\cdot 2 = 16 \text{ and } 8\theta = 8\cdot\frac{\pi}{4} = 2\pi.$$

This means: $(1 + i)^8$ has absolute value 16 and angle 2π. *The eighth power of* $1 + i$ *is the real number* 16.

Powers are easy in polar form. So is multiplication of complex numbers.

The polar form of z^n has absolute value r^n. The angle is n times θ:

$$\textbf{\textit{The nth power of}} \quad z = r(\cos\theta + i\sin\theta) \quad \textbf{\textit{is}} \quad z^n = r^n(\cos n\theta + i\sin n\theta). \tag{3}$$

In that case z multiplies itself. In all cases, ***multiply r's and add the angles***:

$$r(\cos\theta + i\sin\theta) \text{ times } r'(\cos\theta' + i\sin\theta') = rr'\big(\cos(\theta+\theta') + i\sin(\theta+\theta')\big). \tag{4}$$

One way to understand this is by trigonometry. Concentrate on angles. Why do we get the double angle 2θ for z^2?

$$(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = \cos^2\theta + i^2\sin^2\theta + 2i\sin\theta\cos\theta.$$

The real part $\cos^2\theta - \sin^2\theta$ is $\cos 2\theta$. The imaginary part $2\sin\theta\cos\theta$ is $\sin 2\theta$. Those are the "double angle" formulas. They show that θ in z becomes 2θ in z^2.

There is a second way to understand the rule for z^n. It uses the only amazing formula in this section. Remember that $\cos\theta + i\sin\theta$ has absolute value 1. The cosine is made up of even powers, starting with $1 - \frac{1}{2}\theta^2$. The sine is made up of odd powers, starting with $\theta - \frac{1}{6}\theta^3$. The beautiful fact is that $e^{i\theta}$ combines both of those series into $\cos\theta + i\sin\theta$:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \quad \text{becomes} \quad e^{i\theta} = 1 + i\theta + \frac{1}{2}i^2\theta^2 + \frac{1}{6}i^3\theta^3 + \cdots$$

Write -1 for i^2 to see $1 - \frac{1}{2}\theta^2$. ***The complex number*** $e^{i\theta}$ ***is*** $\cos\theta + i\sin\theta$:

$$\textbf{\textit{Euler's Formula}} \quad e^{i\theta} = \cos\theta + i\sin\theta \quad \text{gives} \quad z = r\cos\theta + ir\sin\theta = re^{i\theta} \tag{5}$$

The special choice $\theta = 2\pi$ gives $\cos 2\pi + i\sin 2\pi$ which is 1. Somehow the infinite series $e^{2\pi i} = 1 + 2\pi i + \frac{1}{2}(2\pi i)^2 + \cdots$ adds up to 1.

Now multiply $e^{i\theta}$ times $e^{i\theta'}$. Angles add for the same reason that exponents add:

$$e^2 \text{ times } e^3 \text{ is } e^5 \qquad e^{i\theta} \text{ times } e^{i\theta} \text{ is } e^{2i\theta} \qquad e^{i\theta} \text{ times } e^{i\theta'} \text{ is } e^{i(\theta+\theta')}$$

The powers $(re^{i\theta})^n$ are equal to $r^n e^{in\theta}$. They stay on the unit circle when $r = 1$ and $r^n = 1$. Then we find n different numbers whose nth powers equal 1:

$$\textbf{\textit{Set}} \quad w = e^{2\pi i/n}. \quad \textbf{\textit{The nth powers of}} \ 1, w, w^2, \ldots, w^{n-1} \ \textbf{\textit{all equal 1.}}$$

Those are the "nth roots of 1." They solve the equation $z^n = 1$. They are equally spaced around the unit circle in Figure 10.2b, where the full 2π is divided by n. Multiply their angles by n to take nth powers. That gives $w^n = e^{2\pi i}$ which is 1. Also $(w^2)^n = e^{4\pi i} = 1$. Each of those numbers, to the nth power, comes around the unit circle to 1.

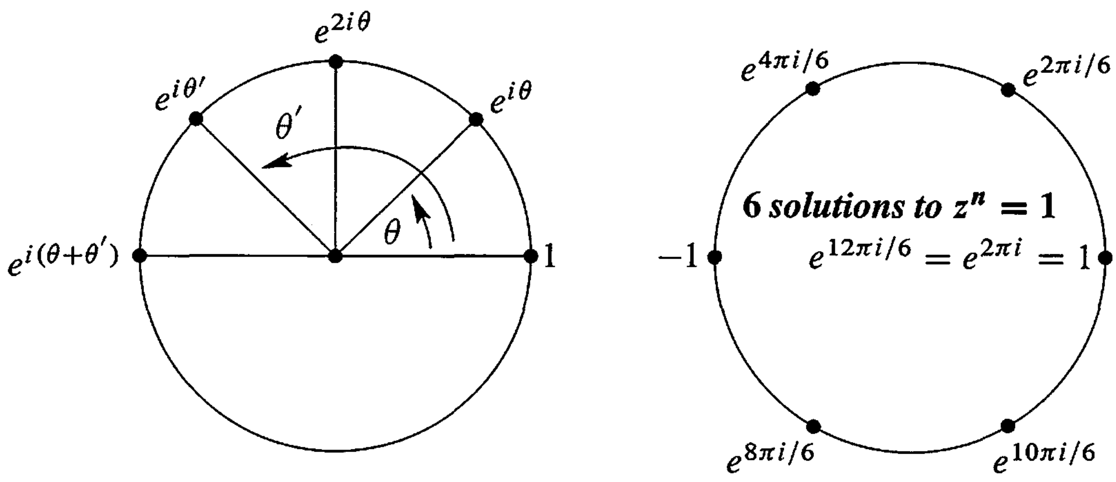


Figure 10.2: (a) Multiplying $e^{i\theta}$ times $e^{i\theta'}$. (b) The nth power of $e^{2\pi i/n}$ is $e^{2\pi i} = 1$.

These n roots of 1 are the key numbers for signal processing. The Discrete Fourier Transform uses w and its powers. Section 10.3 shows how to decompose a vector (a signal) into n frequencies by the Fast Fourier Transform.

■ REVIEW OF THE KEY IDEAS ■

1. Adding $a + ib$ to $c + id$ is like adding $(a, b) + (c, d)$. Use $i^2 = -1$ to multiply.
2. The conjugate of $z = a + bi = re^{i\theta}$ is $\bar{z} = z^* = a - bi = re^{-i\theta}$.
3. z times $\bar{z}$ is $re^{i\theta}$ times $re^{-i\theta}$. This is $r^2 = |z|^2 = a^2 + b^2$ (real).
4. Powers and products are easy in polar form $z = re^{i\theta}$. *Multiply r's and add θ's.*

Problem Set 10.1

Questions 1–8 are about operations on complex numbers.

1 Add and multiply each pair of complex numbers:

(a) $2+i, 2-i$ (b) $-1+i, -1+i$ (c) $\cos\theta + i\sin\theta, \cos\theta - i\sin\theta$

2 Locate these points on the complex plane. Simplify them if necessary:

(a) $2+i$ (b) $(2+i)^2$ (c) $\frac{1}{2+i}$ (d) $|2+i|$

3 Find the absolute value $r = |z|$ of these four numbers. If θ is the angle for $6-8i$, what are the angles for the other three numbers?

(a) $6-8i$ (b) $(6-8i)^2$ (c) $\frac{1}{6-8i}$ (d) $(6+8i)^2$

4 If $|z| = 2$ and $|w| = 3$ then $|z \times w| =$ ____ and $|z + w| \leq$ ____ and $|z/w| =$ ____ and $|z - w| \leq$ ____.

5 Find $a + ib$ for the numbers at angles $30°, 60°, 90°, 120°$ on the unit circle. If w is the number at $30°$, check that w^2 is at $60°$. What power of w equals 1?

6 If $z = r\cos\theta + ir\sin\theta$ then $1/z$ has absolute value ____ and angle ____. Its polar form is ____. Multiply $z \times 1/z$ to get 1.

7 The complex multiplication $M = (a + bi)(c + di)$ is a 2 by 2 real multiplication

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

The right side contains the real and imaginary parts of M. Test $M = (1+3i)(1-3i)$.

8 $A = A_1 + iA_2$ is a complex n by n matrix and $\boldsymbol{b} = \boldsymbol{b}_1 + i\boldsymbol{b}_2$ is a complex vector. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x}_1 + i\boldsymbol{x}_2$. Write $A\boldsymbol{x} = \boldsymbol{b}$ as a real system of size $2n$:

Complex n by n
Real $2n$ by $2n$

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}.$$

Questions 9–16 are about the conjugate $\bar{z} = a - ib = re^{-i\theta} = z^*$.

9 Write down the complex conjugate of each number by changing i to $-i$:

(a) $2 - i$ (b) $(2 - i)(1 - i)$ (c) $e^{i\pi/2}$ (which is i)
(d) $e^{i\pi} = -1$ (e) $\frac{1+i}{1-i}$ (which is also i) (f) $i^{103} =$ ____.

10 The sum $z + \bar{z}$ is always ____. The difference $z - \bar{z}$ is always ____. Assume $z \neq 0$. The product $z \times \bar{z}$ is always ____. The ratio $z/\bar{z}$ always has absolute value ____.

11 For a real matrix, the conjugate of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is $A\bar{\boldsymbol{x}} = \bar{\lambda}\bar{\boldsymbol{x}}$. This proves two things: $\bar{\lambda}$ is another eigenvalue and $\bar{\boldsymbol{x}}$ is its eigenvector. Find the eigenvalues $\lambda, \bar{\lambda}$ and eigenvectors $\boldsymbol{x}, \bar{\boldsymbol{x}}$ of $A = [a \;\; b; \;\; -b \;\; a]$.

12 The eigenvalues of a real 2 by 2 matrix come from the quadratic formula:

$$\det\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

gives the two eigenvalues $\lambda = \left[a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}\right]/2$.

(a) If $a = b = d = 1$, the eigenvalues are complex when c is ____.

(b) What are the eigenvalues when $ad = bc$?

(c) The two eigenvalues (plus sign and minus sign) are not always conjugates of each other. Why not?

13 In Problem 12 the eigenvalues are not real when $(\text{trace})^2 = (a+d)^2$ is smaller than ____. Show that the λ's *are* real when $bc > 0$.

14 Find the eigenvalues and eigenvectors of this permutation matrix:

$$P_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{has} \quad \det(P_4 - \lambda I) = \_\_\_\_ .$$

15 Extend P_4 above to P_6 (five 1's below the diagonal and one in the corner). Find $\det(P_6 - \lambda I)$ and the six eigenvalues in the complex plane.

16 A real skew-symmetric matrix ($A^{\mathrm{T}} = -A$) has pure imaginary eigenvalues. First proof: If $Ax = \lambda x$ then block multiplication gives

$$\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ ix \end{bmatrix} = i\lambda \begin{bmatrix} x \\ ix \end{bmatrix}.$$

This block matrix is symmetric. Its eigenvalues must be ____! So λ is ____.

Questions 17–24 are about the form $re^{i\theta}$ of the complex number $r\cos\theta + ir\sin\theta$.

17 Write these numbers in Euler's form $re^{i\theta}$. Then square each number:

(a) $1 + \sqrt{3}i$ (b) $\cos 2\theta + i \sin 2\theta$ (c) $-7i$ (d) $5 - 5i$.

18 Find the absolute value and the angle for $z = \sin\theta + i\cos\theta$ (careful). Locate this z in the complex plane. Multiply z by $\cos\theta + i\sin\theta$ to get ____.

19 Draw all eight solutions of $z^8 = 1$ in the complex plane. What is the rectangular form $a + ib$ of the root $z = \overline{w} = \exp(-2\pi i/8)$?

20 Locate the cube roots of 1 in the complex plane. Locate the cube roots of -1. Together these are the sixth roots of ____.

21 By comparing $e^{3i\theta} = \cos 3\theta + i\sin 3\theta$ with $(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$, find the "triple angle" formulas for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

22 Suppose the conjugate $\overline{z}$ is equal to the reciprocal $1/z$. What are all possible z's?

23 (a) Why do e^i and i^e both have absolute value 1?

(b) In the complex plane put stars near the points e^i and i^e.

(c) The number i^e could be $(e^{i\pi/2})^e$ or $(e^{5i\pi/2})^e$. Are those equal?

24 Draw the paths of these numbers from $t = 0$ to $t = 2\pi$ in the complex plane:

(a) e^{it} (b) $e^{(-1+i)t} = e^{-t}e^{it}$ (c) $(-1)^t = e^{t\pi i}$.

10.2 Hermitian and Unitary Matrices

The main message of this section can be presented in one sentence: ***When you transpose a complex vector z or matrix A, take the complex conjugate too.*** Don't stop at z^{T} or A^{T}. Reverse the signs of all imaginary parts. From a column vector with $z_j = a_j + ib_j$, the good row vector is the *conjugate transpose* with components $a_j - ib_j$:

Conjugate transpose $$\overline{z}^{\mathrm{T}} = \begin{bmatrix} \overline{z}_1 & \cdots & \overline{z}_n \end{bmatrix} = \begin{bmatrix} a_1 - ib_1 & \cdots & a_n - ib_n \end{bmatrix}. \qquad (1)$$

Here is one reason to go to $\overline{z}$. The length squared of a real vector is $x_1^2 + \cdots + x_n^2$. The length squared of a complex vector is *not* $z_1^2 + \cdots + z_n^2$. With that wrong definition, the length of $(1, i)$ would be $1^2 + i^2 = 0$. A nonzero vector would have zero length—not good. Other vectors would have complex lengths. Instead of $(a + bi)^2$ we want $a^2 + b^2$, the *absolute value squared.* This is $(a + bi)$ times $(a - bi)$.

For each component we want z_j times $\overline{z}_j$, which is $|z_j|^2 = a_j^2 + b_j^2$. That comes when the components of z multiply the components of $\overline{z}$:

Length squared $$\begin{bmatrix} \overline{z}_1 & \cdots & \overline{z}_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = |z_1|^2 + \cdots + |z_n|^2. \quad \textbf{This is} \quad \overline{z}^{\mathrm{T}} z = \|z\|^2. \qquad (2)$$

Now the squared length of $(1, i)$ is $1^2 + |i|^2 = 2$. The length is $\sqrt{2}$. The squared length of $(1 + i, 1 - i)$ is 4. The only vectors with zero length are zero vectors.

The length $\|z\|$ is the square root of $\overline{z}^{\mathrm{T}} z = z^{\mathrm{H}} z = |z_1|^2 + \cdots + |z_n|^2$

Before going further we replace two symbols by one symbol. Instead of a bar for the conjugate and T for the transpose, we just use a superscript H. Thus $\overline{z}^{\mathrm{T}} = z^{\mathrm{H}}$. This is "$z$ Hermitian," the *conjugate transpose* of z. The new word is pronounced "Hermeeshan." The new symbol applies also to matrices: The conjugate transpose of a matrix A is A^{H}.

Another popular notation is A^*. The MATLAB transpose command $'$ automatically takes complex conjugates (A' is A^{H}).

The vector z^{H} is $\overline{z}^{\mathrm{T}}$. The matrix A^{H} is $\overline{A}^{\mathrm{T}}$, the conjugate transpose of A:

$$A^{\mathrm{H}} = \text{"A Hermitian"} \quad \text{If} \quad A = \begin{bmatrix} 1 & i \\ 0 & 1+i \end{bmatrix} \quad \text{then} \quad A^{\mathrm{H}} = \begin{bmatrix} 1 & 0 \\ -i & 1-i \end{bmatrix}$$

Complex Inner Products

For real vectors, the length squared is $x^{\mathrm{T}}x$—*the inner product of x with itself.* For complex vectors, the length squared is $z^{\mathrm{H}}z$. It will be very desirable if $z^{\mathrm{H}}z$ is the inner product of z with itself. To make that happen, the complex inner product should use the conjugate transpose (not just the transpose). The inner product sees no change when the vectors are real, but there is a definite effect from choosing $\overline{u}^{\mathrm{T}}$, when u is complex:

DEFINITION The inner product of real or complex vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$:

$$\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = \begin{bmatrix}\overline{u}_1 & \cdots & \overline{u}_n\end{bmatrix}\begin{bmatrix}v_1\\ \vdots \\ v_n\end{bmatrix} = \overline{u}_1 v_1 + \cdots + \overline{u}_n v_n. \tag{3}$$

With complex vectors, $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$ is different from $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u}$. *The order of the vectors is now important.* In fact $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u} = \overline{v}_1 u_1 + \cdots + \overline{v}_n u_n$ is the complex conjugate of $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$. We have to put up with a few inconveniences for the greater good.

Example 1 The inner product of $\boldsymbol{u} = \begin{bmatrix}1\\ i\end{bmatrix}$ with $\boldsymbol{v} = \begin{bmatrix}i\\ 1\end{bmatrix}$ is $\begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix}i\\ 1\end{bmatrix} = 0$.

Example 1 is surprising. Those vectors $(1, i)$ and $(i, 1)$ don't look perpendicular. But they are. ***A zero inner product still means that the*** (complex) ***vectors are orthogonal.*** Similarly the vector $(1, i)$ is orthogonal to the vector $(1, -i)$. Their inner product is $1 - 1 = 0$. We are correctly getting zero for the inner product—where we would be incorrectly getting zero for the length of $(1, i)$ if we forgot to take the conjugate.

Note We have chosen to conjugate the first vector $\boldsymbol{u}$. Some authors choose the second vector $\boldsymbol{v}$. Their complex inner product would be $\boldsymbol{u}^{\mathrm{T}}\overline{\boldsymbol{v}}$. It is a free choice, as long as we stick to it. We wanted to use the single symbol $^{\mathrm{H}}$ in the next formula too:

The inner product of $A\boldsymbol{u}$ with $\boldsymbol{v}$ equals the inner product of $\boldsymbol{u}$ with $A^{\mathrm{H}}\boldsymbol{v}$:

$$A^{\mathrm{H}} = \text{“adjoint” of } A \qquad (A\boldsymbol{u})^{\mathrm{H}}\boldsymbol{v} = \boldsymbol{u}^{\mathrm{H}}(A^{\mathrm{H}}\boldsymbol{v}). \tag{4}$$

The conjugate of $A\boldsymbol{u}$ is $\overline{A\boldsymbol{u}}$. Transposing it gives $\overline{\boldsymbol{u}}^{\mathrm{T}}\overline{A}^{\mathrm{T}}$ as usual. This is $\boldsymbol{u}^{\mathrm{H}}A^{\mathrm{H}}$. Everything that should work, does work. The rule for $^{\mathrm{H}}$ comes from the rule for $^{\mathrm{T}}$. That applies to products of matrices:

The conjugate transpose of AB is $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$.

We constantly use the fact that $(a - ib)(c - id)$ is the conjugate of $(a + ib)(c + id)$.

Hermitian Matrices

Among real matrices, the *symmetric matrices* form the most important special class: $A = A^{\mathrm{T}}$. They have real eigenvalues and a full set of orthogonal eigenvectors. The diagonalizing matrix S is an orthogonal matrix Q. Every symmetric matrix can be written as $A = Q\Lambda Q^{-1}$ and also as $A = Q\Lambda Q^{\mathrm{T}}$ (because $Q^{-1} = Q^{\mathrm{T}}$). All this follows from $a_{ij} = a_{ji}$, when A is real.

Among complex matrices, the special class contains the ***Hermitian matrices***: $A = A^{\mathrm{H}}$. The condition on the entries is $a_{ij} = \overline{a_{ji}}$. In this case we say that "*A is* Hermitian." *Every real symmetric matrix is Hermitian*, because taking its conjugate has no effect. The next matrix is also Hermitian, $A = A^{\mathrm{H}}$:

Example 2 $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ The main diagonal is real since $a_{ii} = \overline{a_{ii}}$. Across it are conjugates $3+3i$ and $3-3i$.

This example will illustrate the three crucial properties of all Hermitian matrices.

If $A = A^{\mathrm{H}}$ and z is any vector, the number $z^{\mathrm{H}}Az$ is real.

Quick proof: $z^{\mathrm{H}}Az$ is certainly 1 by 1. Take its conjugate transpose:

$$(z^{\mathrm{H}}Az)^{\mathrm{H}} = z^{\mathrm{H}}A^{\mathrm{H}}(z^{\mathrm{H}})^{\mathrm{H}} \quad \text{which is } z^{\mathrm{H}}Az \text{ again.}$$

This used $A = A^{\mathrm{H}}$. So the number $z^{\mathrm{H}}Az$ equals its conjugate and must be real. Here is that "energy" $z^{\mathrm{H}}Az$ in our example:

$$\begin{bmatrix} \overline{z}_1 & \overline{z}_2 \end{bmatrix} \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{2\overline{z}_1 z_1 + 5\overline{z}_2 z_2}_{\textit{diagonal}} + \underbrace{(3-3i)\overline{z}_1 z_2 + (3+3i) z_1 \overline{z}_2}_{\textit{off-diagonal}}.$$

The terms $2|z_1|^2$ and $5|z_2|^2$ from the diagonal are both real. The off-diagonal terms are conjugates of each other—so their sum is real. (The imaginary parts cancel when we add.) The whole expression $z^{\mathrm{H}}Az$ is real, and this will make λ real.

Every eigenvalue of a Hermitian matrix is real.

Proof Suppose $Az = \lambda z$. *Multiply both sides by z^{H} to get $z^{\mathrm{H}}Az = \lambda z^{\mathrm{H}}z$.* On the left side, $z^{\mathrm{H}}Az$ is real. On the right side, $z^{\mathrm{H}}z$ is the length squared, real and positive. So the ratio $\lambda = z^{\mathrm{H}}Az / z^{\mathrm{H}}z$ is a real number. Q.E.D.

The example above has eigenvalues $\lambda = 8$ and $\lambda = -1$, real because $A = A^{\mathrm{H}}$:

$$\begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 - |3+3i|^2$$
$$= \lambda^2 - 7\lambda + 10 - 18 = (\lambda - 8)(\lambda + 1).$$

The eigenvectors of a Hermitian matrix are orthogonal (when they correspond to different eigenvalues). If $Az = \lambda z$ and $Ay = \beta y$ then $y^{\mathrm{H}}z = 0$.

Proof Multiply $Az = \lambda z$ on the left by y^{H}. Multiply $y^{\mathrm{H}}A^{\mathrm{H}} = \beta y^{\mathrm{H}}$ on the right by z:

$$y^{\mathrm{H}}Az = \lambda y^{\mathrm{H}}z \quad \text{and} \quad y^{\mathrm{H}}A^{\mathrm{H}}z = \beta y^{\mathrm{H}}z. \tag{5}$$

The left sides are equal because $A = A^{\mathrm{H}}$. Therefore the right sides are equal. Since β is different from λ, the other factor $y^{\mathrm{H}}z$ must be zero. The eigenvectors are orthogonal, as in our example with $\lambda = 8$ and $\beta = -1$:

$$(A - 8I)z = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$(A + I)y = \begin{bmatrix} 3 & 3-3i \\ 3+3i & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}.$$

Take the inner product of those eigenvectors y and z:

Orthogonal eigenvectors $$y^{\mathrm{H}}z = \begin{bmatrix} 1+i & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = 0.$$

These eigenvectors have squared length $1^2 + 1^2 + 1^2 = 3$. After division by $\sqrt{3}$ they are unit vectors. They were orthogonal, now they are ***orthonormal.*** They go into the columns of the *eigenvector matrix* S, which diagonalizes A.

When A is real and symmetric, S is Q—an orthogonal matrix. Now A is complex and Hermitian. Its eigenvectors are complex and orthonormal. ***The eigenvector matrix S is like Q, but complex.*** We now assign a new name "unitary" and a new letter U to a complex orthogonal matrix.

Unitary Matrices

A ***unitary matrix*** U is a (complex) square matrix that has ***orthonormal columns.*** U is the complex equivalent of Q. The eigenvectors of A give a perfect example:

Unitary matrix $$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

This U is also a Hermitian matrix. I didn't expect that! The example is almost too perfect. We will see that the eigenvalues of this U must be 1 and -1.

The matrix test for real orthonormal columns was $Q^{\mathrm{T}}Q = I$. When Q^{T} multiplies Q, the zero inner products appear off the diagonal. In the complex case, Q becomes U. The columns show themselves as orthonormal when U^{H} multiplies U. The inner products of the columns are again 1 and 0. They fill up $\boldsymbol{U^{\mathrm{H}}U = I}$:

Every matrix U with orthonormal columns has $\boldsymbol{U^{\mathrm{H}}U = I}$.

If U is square, it is a unitary matrix. Then $\boldsymbol{U^{\mathrm{H}} = U^{-1}}$.

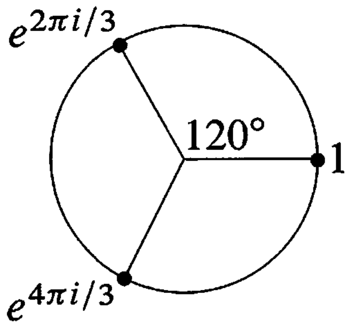
Suppose U (with orthonormal columns) multiplies any z. The vector length stays the same, because $z^{\mathrm{H}}U^{\mathrm{H}}Uz = z^{\mathrm{H}}z$. If z is an eigenvector of U we learn something more: ***The eigenvalues of unitary (and orthogonal) matrices all have absolute value*** $|\lambda| = 1$.

If U is unitary then $\|Uz\| = \|z\|$. Therefore $Uz = \lambda z$ leads to $|\lambda| = 1$.

Our 2 by 2 example is both Hermitian ($U = U^{\mathrm{H}}$) and unitary ($U^{-1} = U^{\mathrm{H}}$). That means real eigenvalues ($\lambda = \overline{\lambda}$), and it means $|\lambda| = 1$. A real number with absolute value 1 has only two possibilities: *The eigenvalues are* 1 *or* -1.

Since the trace is zero for our U, one eigenvalue is $\lambda = 1$ and the other is $\lambda = -1$.

Example 3 The 3 by 3 ***Fourier matrix*** is in Figure 10.3. Is it Hermitian? Is it unitary? F_3 is certainly symmetric. It equals its transpose. But it doesn't equal its conjugate transpose—*it is not Hermitian.* If you change i to $-i$, you get a different matrix.



Fourier matrix $$F = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}.$$

Figure 10.3: The cube roots of 1 go into the Fourier matrix $F = F_3$.

Is F unitary? *Yes.* The squared length of every column is $\frac{1}{3}(1+1+1)$ (unit vector). The first column is orthogonal to the second column because $1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$. This is the sum of the three numbers marked in Figure 10.3.

Notice the symmetry of the figure. If you rotate it by 120°, the three points are in the same position. Therefore their sum S also stays in the same position! The only possible sum in the same position after 120° rotation is $S = 0$.

Is column 2 of F orthogonal to column 3? Their dot product looks like

$$\tfrac{1}{3}(1 + e^{6\pi i/3} + e^{6\pi i/3}) = \tfrac{1}{3}(1+1+1).$$

This is not zero. The answer is wrong because we forgot to take complex conjugates. The complex inner product uses $^{\mathrm{H}}$ not $^{\mathrm{T}}$:

$$\begin{aligned}(\text{column } 2)^{\mathrm{H}}(\text{column } 3) &= \tfrac{1}{3}(1\cdot 1 + e^{-2\pi i/3}e^{4\pi i/3} + e^{-4\pi i/3}e^{2\pi i/3}) \\ &= \tfrac{1}{3}(1 + e^{2\pi i/3} + e^{-2\pi i/3}) = 0.\end{aligned}$$

So we do have orthogonality. ***Conclusion: F is a unitary matrix.***

The next section will study the n by n Fourier matrices. Among all complex unitary matrices, these are the most important. When we multiply a vector by F, we are computing its ***Discrete Fourier Transform***. When we multiply by F^{-1}, we are computing the *inverse transform*. The special property of unitary matrices is that $F^{-1} = F^{\mathrm{H}}$. The inverse transform only differs by changing i to $-i$:

Change i to $-i$ $$F^{-1} = F^{\mathrm{H}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{-4\pi i/3} \\ 1 & e^{-4\pi i/3} & e^{-2\pi i/3} \end{bmatrix}.$$

Everyone who works with F recognizes its value. The last section of the book will bring together Fourier analysis and complex numbers and linear algebra.

This section ends with a table to translate between real and complex—for vectors and for matrices:

Real versus Complex

$\mathbf{R}^n$: vectors with n real components	$\leftrightarrow$	$\mathbf{C}^n$: vectors with n complex components
length: $\|x\|^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	length: $\|z\|^2 = \lvert z_1\rvert^2 + \cdots + \lvert z_n\rvert^2$
transpose: $(A^{\mathrm{T}})_{ij} = A_{ji}$	$\leftrightarrow$	conjugate transpose: $(A^{\mathrm{H}})_{ij} = \overline{A_{ji}}$
product rule: $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	$\leftrightarrow$	product rule: $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
dot product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	$\leftrightarrow$	inner product: $u^{\mathrm{H}}v = \overline{u}_1v_1 + \cdots + \overline{u}_nv_n$
reason for A^{T}: $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	$\leftrightarrow$	reason for A^{H}: $(Au)^{\mathrm{H}}v = u^{\mathrm{H}}(A^{\mathrm{H}}v)$
orthogonality: $x^{\mathrm{T}}y = 0$	$\leftrightarrow$	orthogonality: $u^{\mathrm{H}}v = 0$
symmetric matrices: $A = A^{\mathrm{T}}$	$\leftrightarrow$	Hermitian matrices: $A = A^{\mathrm{H}}$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$(real Λ)	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric matrices: $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian matrices $K^{\mathrm{H}} = -K$
orthogonal matrices: $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary matrices: $U^{\mathrm{H}} = U^{-1}$
orthonormal columns: $Q^{\mathrm{T}}Q = I$	$\leftrightarrow$	orthonormal columns: $U^{\mathrm{H}}U = I$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|Qx\| = \|x\|$	$\leftrightarrow$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|Uz\| = \|z\|$

The columns and also the eigenvectors of Q and U are orthonormal. Every $|\lambda| = 1$.

Problem Set 10.2

1 Find the lengths of $u = (1+i, 1-i, 1+2i)$ and $v = (i, i, i)$. Also find $u^{\mathrm{H}}v$ and $v^{\mathrm{H}}u$.

2 Compute $A^{\mathrm{H}}A$ and AA^{H}. Those are both _____ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

3 Solve $Az = \mathbf{0}$ to find a vector in the nullspace of A in Problem 2. Show that z is orthogonal to the columns of A^{H}. Show that z is *not* orthogonal to the columns of A^{T}. ***The good row space is no longer $C(A^{\mathrm{T}})$.*** Now it is $C(A^{\mathrm{H}})$.

4 Problem 3 indicates that the four fundamental subspaces are $C(A)$ and $N(A)$ and _____ and _____. Their dimensions are still r and $n-r$ and r and $m-r$. They are still orthogonal subspaces. *The symbol* $^{\mathrm{H}}$ *takes the place of* $^{\mathrm{T}}$.

5 (a) Prove that $A^{\mathrm{H}}A$ is always a Hermitian matrix.

(b) If $Az = \mathbf{0}$ then $A^{\mathrm{H}}Az = \mathbf{0}$. If $A^{\mathrm{H}}Az = \mathbf{0}$, multiply by z^{H} to prove that $Az = \mathbf{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are _____. Therefore $A^{\mathrm{H}}A$ is an invertible Hermitian matrix when the nullspace of A contains only $z = 0$.

6 True or false (give a reason if true or a counterexample if false):

(a) If A is a real matrix then $A + iI$ is invertible.

(b) If A is a Hermitian matrix then $A + iI$ is invertible.

(c) If U is a unitary matrix then $A + iI$ is invertible.

7 When you multiply a Hermitian matrix by a real number c, is cA still Hermitian? Show that iA is skew-Hermitian when A is Hermitian. The 3 by 3 Hermitian matrices are a subspace provided the "scalars" are real numbers.

8 Which classes of matrices does P belong to: invertible, Hermitian, unitary?

$$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}.$$

Compute P^2, P^3, and P^{100}. What are the eigenvalues of P?

9 Find the unit eigenvectors of P in Problem 8, and put them into the columns of a unitary matrix F. What property of P makes these eigenvectors orthogonal?

10 Write down the 3 by 3 circulant matrix $C = 2I + 5P$. It has the same eigenvectors as P in Problem 8. Find its eigenvalues.

11 If U and V are unitary matrices, show that U^{-1} is unitary and also UV is unitary. Start from $U^{\mathrm{H}}U = I$ and $V^{\mathrm{H}}V = I$.

12 How do you know that the determinant of every Hermitian matrix is real?

13 The matrix $A^{\mathrm{H}}A$ is not only Hermitian but also positive definite, when the columns of A are independent. Proof: $z^{\mathrm{H}}A^{\mathrm{H}}Az$ is positive if z is nonzero because _____.

14 Diagonalize this Hermitian matrix to reach $A = U\Lambda U^{\mathrm{H}}$:

$$A = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix}.$$

15 Diagonalize this skew-Hermitian matrix to reach $K = U\Lambda U^{\mathrm{H}}$. All λ's are ____:

$$K = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}.$$

16 Diagonalize this orthogonal matrix to reach $Q = U\Lambda U^{\mathrm{H}}$. Now all λ's are ____:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

17 Diagonalize this unitary matrix V to reach $V = U\Lambda U^{\mathrm{H}}$. Again all λ's are ____:

$$V = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}.$$

18 If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthonormal basis for $\mathbf{C}^n$, the matrix with those columns is a ____ matrix. Show that any vector $\boldsymbol{z}$ equals $(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z})\boldsymbol{v}_1 + \cdots + (\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})\boldsymbol{v}_n$.

19 The functions e^{-ix} and e^{ix} are orthogonal on the interval $0 \le x \le 2\pi$ because their inner product is $\int_0^{2\pi}$ ____ $= 0$.

20 The vectors $\boldsymbol{v} = (1, i, 1)$, $\boldsymbol{w} = (i, 1, 0)$ and $\boldsymbol{z} =$ ____ are an orthogonal basis for ____.

21 If $A = R + iS$ is a Hermitian matrix, are its real and imaginary parts symmetric?

22 The (complex) dimension of $\mathbf{C}^n$ is ____. Find a non-real basis for $\mathbf{C}^n$.

23 Describe all 1 by 1 and 2 by 2 Hermitian matrices and unitary matrices.

24 How are the eigenvalues of A^{H} related to the eigenvalues of the square complex matrix A?

25 If $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{u} = 1$ show that $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is Hermitian and also unitary. The rank-one matrix $\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is the projection onto what line in $\mathbf{C}^n$?

26 If $A + iB$ is a unitary matrix (A and B are real) show that $Q = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is an orthogonal matrix.

27 If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric.

28 Prove that the inverse of a Hermitian matrix is also Hermitian (transpose $A^{-1}A = I$).

29 Diagonalize this matrix by constructing its eigenvalue matrix Λ and its eigenvector matrix S:

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} = A^{\mathrm{H}}.$$

30 A matrix with orthonormal eigenvectors has the form $A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$. *Prove that* $AA^{\mathrm{H}} = A^{\mathrm{H}}A$. These are exactly the **normal matrices**. Examples are Hermitian, skew-Hermitian, and unitary matrices. Construct a 2 by 2 normal matrix by choosing complex eigenvalues in Λ.

10.3 The Fast Fourier Transform

Many applications of linear algebra take time to develop. It is not easy to explain them in an hour. The teacher and the author must choose between completing the theory and adding new applications. Often the theory wins, but this section is an exception. It explains the most valuable numerical algorithm in the last century.

We want to multiply quickly by F and F^{-1}, the Fourier matrix and its inverse. This is achieved by the Fast Fourier Transform. An ordinary product Fc uses n^2 multiplications (F has n^2 entries). The FFT needs only n times $\frac{1}{2}\log_2 n$. We will see how.

The FFT has revolutionized signal processing. Whole industries are speeded up by this one idea. Electrical engineers are the first to know the difference—they take your Fourier transform as they meet you (if you are a function). Fourier's idea is to represent f as a sum of harmonics $c_k e^{ikx}$. The function is seen in *frequency space* through the coefficients c_k, instead of *physical space* through its values $f(x)$. The passage backward and forward between c's and f's is by the Fourier transform. Fast passage is by the FFT.

Roots of Unity and the Fourier Matrix

Quadratic equations have two roots (or one repeated root). Equations of degree n have n roots (counting repetitions). This is the Fundamental Theorem of Algebra, and to make it true we must allow complex roots. This section is about the very special equation $z^n = 1$. The solutions z are the "nth roots of unity." They are n evenly spaced points around the unit circle in the complex plane.

Figure 10.4 shows the eight solutions to $z^8 = 1$. Their spacing is $\frac{1}{8}(360°) = 45°$. The first root is at 45° or $\theta = 2\pi/8$ radians. ***It is the complex number $w = e^{i\theta} = e^{i2\pi/8}$.*** We call this number w_8 to emphasize that it is an 8th root. You could write it in terms of $\cos\frac{2\pi}{8}$ and $\sin\frac{2\pi}{8}$, but don't do it. The seven other 8th roots are $w^2, w^3, \ldots, w^8$, going around the circle. Powers of w are best in polar form, because we work only with the angles $\frac{2\pi}{8}, \frac{4\pi}{8}, \ldots, \frac{16\pi}{8} = 2\pi$.

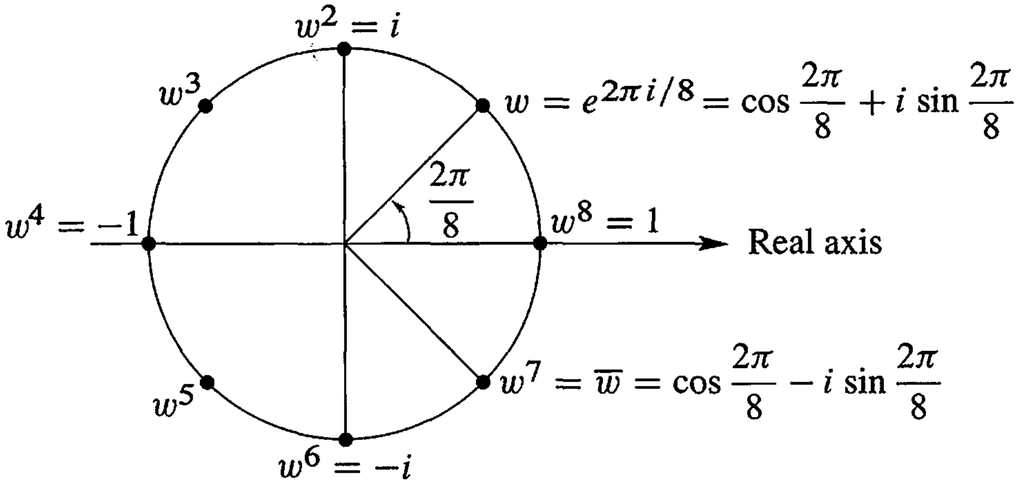


Figure 10.4: The eight solutions to $z^8 = 1$ are $1, w, w^2, \ldots, w^7$ with $w = (1+i)/\sqrt{2}$.

The fourth roots of 1 are also in the figure. They are $i, -1, -i, 1$. The angle is now $2\pi/4$ or $90°$. The first root $w_4 = e^{2\pi i/4}$ is nothing but i. Even the square roots of 1 are seen, with $w_2 = e^{i2\pi/2} = -1$. Do not despise those square roots 1 and -1. The idea behind the FFT is to go from an 8 by 8 Fourier matrix (containing powers of w_8) to the 4 by 4 matrix below (with powers of $w_4 = i$). The same idea goes from 4 to 2. By exploiting the connections of F_8 down to F_4 and up to F_{16} (and beyond), the FFT makes multiplication by F_{1024} very quick.

We describe the *Fourier matrix*, first for $n = 4$. Its rows contain powers of 1 and w and w^2 and w^3. These are the fourth roots of 1, and their powers come in a special order.

Fourier matrix $n = 4$

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}.$$

The matrix is symmetric ($F = F^{\mathrm{T}}$). It is *not* Hermitian. Its main diagonal is not real. But $\frac{1}{2}F$ is a ***unitary matrix***, which means that $(\frac{1}{2}F^{\mathrm{H}})(\frac{1}{2}F) = I$:

The columns of F give $F^{\mathrm{H}}F = 4I$. ***Its inverse is*** $\frac{1}{4}F^{\mathrm{H}}$ ***which is*** $F^{-1} = \frac{1}{4}\overline{F}$.

The inverse changes from $w = i$ to $\overline{w} = -i$. That takes us from F to $\overline{F}$. When the Fast Fourier Transform gives a quick way to multiply by F, it does the same for F^{-1}.

The unitary matrix is $U = F/\sqrt{n}$. We avoid that $\sqrt{n}$ and just put $\frac{1}{n}$ outside F^{-1}. The main point is to multiply F times the Fourier coefficients c_0, c_1, c_2, c_3:

4-point Fourier series

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = Fc = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{1}$$

The input is four complex coefficients c_0, c_1, c_2, c_3. The output is four function values y_0, y_1, y_2, y_3. The first output $y_0 = c_0 + c_1 + c_2 + c_3$ is the value of the Fourier series at $x = 0$. *The second output is the value of that series* $\sum c_k e^{ikx}$ *at* $x = 2\pi/4$:

$$y_1 = c_0 + c_1 e^{i2\pi/4} + c_2 e^{i4\pi/4} + c_3 e^{i6\pi/4} = c_0 + c_1 w + c_2 w^2 + c_3 w^3.$$

The third and fourth outputs y_2 and y_3 are the values of $\sum c_k e^{ikx}$ at $x = 4\pi/4$ and $x = 6\pi/4$. These are *finite* Fourier series! They contain $n = 4$ terms and they are evaluated at $n = 4$ points. Those points $x = 0, 2\pi/4, 4\pi/4, 6\pi/4$ are equally spaced.

The next point would be $x = 8\pi/4$ which is 2π. Then the series is back to y_0, because $e^{2\pi i}$ is the same as $e^0 = 1$. Everything cycles around with period 4. In this world $2 + 2$ is 0 because $(w^2)(w^2) = w^0 = 1$. We will follow the convention that j *and* k *go from* 0 *to* $n - 1$ (instead of 1 to n). The "zeroth row" and "zeroth column" of F contain all ones.

The n by n Fourier matrix contains powers of $w = e^{2\pi i/n}$:

$$F_n c = \begin{bmatrix} 1 & 1 & 1 & \cdot & 1 \\ 1 & w & w^2 & \cdot & w^{n-1} \\ 1 & w^2 & w^4 & \cdot & w^{2(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & w^{n-1} & w^{2(n-1)} & \cdot & w^{(n-1)^2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \\ y_{n-1} \end{bmatrix} = y. \qquad (2)$$

F_n is symmetric but not Hermitian. ***Its columns are orthogonal***, and $F_n \overline{F}_n = nI$. *Then* F_n^{-1} *is* $\overline{F}_n/n$. The inverse contains powers of $\overline{w}_n = e^{-2\pi i/n}$. Look at the pattern in F:

The entry in row j, column k is w^{jk}. Row zero and column zero contain $w^0 = 1$.

When we multiply c by F_n, we sum the series at n points. *When we multiply y by F_n^{-1}, we find the coefficients c from the function values y.* In MATLAB that command is c = fft(y). The matrix F passes from "frequency space" to "physical space."

Important note. Many authors prefer to work with $\omega = e^{-2\pi i/N}$, which is the *complex conjugate* of our w. (They often use the Greek omega, and I will do that to keep the two options separate.) With this choice, their DFT matrix contains powers of ω not w. It is conj (F) = complex conjugate of our F. This takes us to frequency space.

$\overline{F}$ is a completely reasonable choice! MATLAB uses $\omega = e^{-2\pi i/N}$. The DFT matrix fft(eye(N)) contains powers of this number $\omega = \overline{w}$. **The Fourier matrix with w's reconstructs y from c.** ***The matrix $\overline{F}$ with ω's computes Fourier coefficients as* fft(y).**

Also important. When a function $f(x)$ has period 2π, and we change x to $e^{i\theta}$, the function is defined around the unit circle (where $z = e^{i\theta}$). Then the Discrete Fourier Transform from y to c is matching n values of this $f(z)$ by a polynomial $p(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}$.

Interpolation Find $c_0, \ldots, c_{n-1}$ so that $p(z) = f(z)$ at n points $z = 1, \ldots, w^{n-1}$

The Fourier matrix is the Vandermonde matrix for interpolation at those n points.

One Step of the Fast Fourier Transform

We want to multiply F times c as quickly as possible. Normally a matrix times a vector takes n^2 separate multiplications—the matrix has n^2 entries. You might think it is impossible to do better. (If the matrix has zero entries then multiplications can be skipped. But the Fourier matrix has no zeros!) By using the special pattern w^{jk} for its entries, F can be factored in a way that produces many zeros. This is the **FFT**.

The key idea is to connect F_n with the half-size Fourier matrix $F_{n/2}$. Assume that n is a power of 2 (say $n = 2^{10} = 1024$). We will connect F_{1024} to F_{512}—or rather to ***two***

copies of F_{512}. When $n = 4$, the key is in the relation between these matrices:

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}.$$

On the left is F_4, with no zeros. On the right is a matrix that is half zero. The work is cut in half. But wait, those matrices are not the same. We need two sparse and simple matrices to complete the FFT factorization:

Factors for FFT
$$F_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}. \tag{3}$$

The last matrix is a permutation. It puts the even c's (c_0 and c_2) ahead of the odd c's (c_1 and c_3). The middle matrix performs half-size transforms F_2 and F_2 on the evens and odds. The matrix at the left combines the two half-size outputs—in a way that produces the correct full-size output $\boldsymbol{y} = F_4\boldsymbol{c}$.

The same idea applies when $n = 1024$ and $m = \frac{1}{2}n = 512$. The number w is $e^{2\pi i/1024}$. It is at the angle $\theta = 2\pi/1024$ on the unit circle. The Fourier matrix F_{1024} is full of powers of w. The first stage of the FFT is the great factorization discovered by Cooley and Tukey (and foreshadowed in 1805 by Gauss):

$$\boldsymbol{F_{1024}} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} \boldsymbol{F_{512}} & \\ & \boldsymbol{F_{512}} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}. \tag{4}$$

I_{512} is the identity matrix. D_{512} is the diagonal matrix with entries $(1, w, \ldots, w^{511})$. The two copies of F_{512} are what we expected. Don't forget that they use the 512th root of unity (which is nothing but w^2!!) The permutation matrix separates the incoming vector $\boldsymbol{c}$ into its even and odd parts $\boldsymbol{c}' = (c_0, c_2, \ldots, c_{1022})$ and $\boldsymbol{c}'' = (c_1, c_3, \ldots, c_{1023})$.

Here are the algebra formulas which say the same thing as the factorization of F_{1024}:

(FFT) Set $m = \frac{1}{2}n$. The first m and last m components of $\boldsymbol{y} = F_n\boldsymbol{c}$ combine the half-size transforms $\boldsymbol{y}' = F_m\boldsymbol{c}'$ and $\boldsymbol{y}'' = F_m\boldsymbol{c}''$. Equation (4) shows this step from n to $m = n/2$ as $I\boldsymbol{y}' + D\boldsymbol{y}''$ and $I\boldsymbol{y}' - D\boldsymbol{y}''$:

$$\begin{aligned} y_j &= y'_j + w_n^j y''_j, \quad j = 0, \ldots, m-1 \\ y_{j+m} &= y'_j - w_n^j y''_j, \quad j = 0, \ldots, m-1. \end{aligned} \tag{5}$$

Split $\boldsymbol{c}$ into $\boldsymbol{c}'$ and $\boldsymbol{c}''$, transform them by F_m into $\boldsymbol{y}'$ and $\boldsymbol{y}''$, and reconstruct $\boldsymbol{y}$.

Those formulas come from separating even c_{2k} from odd c_{2k+1}:

$$y_j = \sum_0^{n-1} w^{jk} c_k = \sum_0^{m-1} w^{2jk} c_{2k} + \sum_0^{m-1} w^{j(2k+1)} c_{2k+1} \text{ with } m = \tfrac{1}{2}n. \tag{6}$$

The even c's go into $c' = (c_0, c_2, \ldots)$ and the odd c's go into $c'' = (c_1, c_3, \ldots)$. Then come the transforms $F_m c'$ and $F_m c''$. The key is $\boldsymbol{w_n^2 = w_m}$. This gives $w_n^{2jk} = w_m^{jk}$.

$$\textbf{Rewrite} \qquad y_j = \sum w_m^{jk} c_k' + (w_n)^j \sum w_m^{jk} c_k'' = y_j' + (w_n)^j y_j''. \tag{7}$$

For $j \geq m$, the minus sign in (5) comes from factoring out $(w_n)^m = -1$.

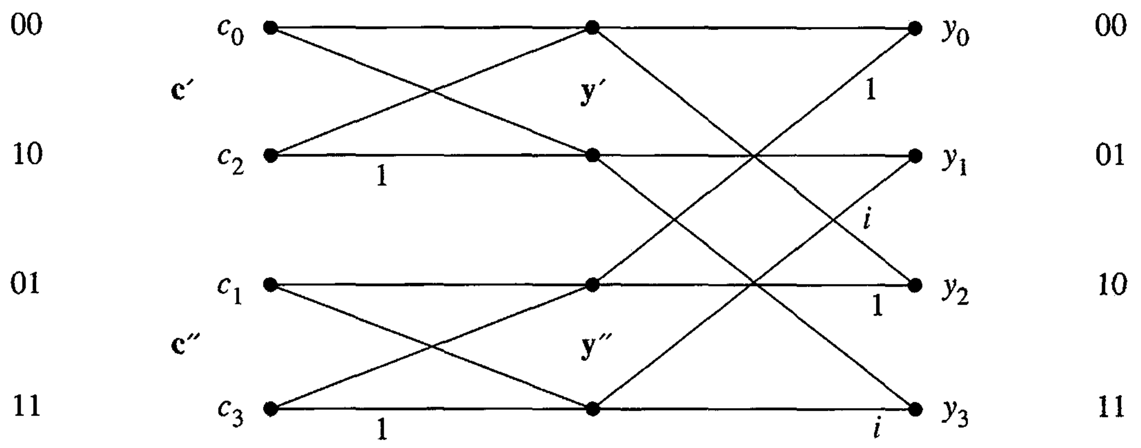
MATLAB easily separates even c's from odd c's and multiplies by w_n^j. We use conj(F) or equivalently MATLAB's inverse transform ifft, because fft is based on $\omega = \overline{w} = e^{-2\pi i/n}$. Problem 17 shows that F and conj(F) are linked by permuting rows.

FFT step from n to $n/2$ in MATLAB

```
y' = ifft (c(0 : 2 : n − 2)) ∗ n/2;
y'' = ifft (c(1 : 2 : n − 1)) ∗ n/2;
d = w.^(0 : n/2 − 1)';
y = [y' + d.∗ y''; y' − d.∗ y''];
```

The flow graph shows c' and c'' going through the half-size F_2. Those steps are called "*butterflies*," from their shape. Then the outputs y' and y'' are combined (multiplying y'' by $1, i$ and also by $-1, -i$) to produce $y = F_4 c$.

This reduction from F_n to two F_m's almost cuts the work in half—you see the zeros in the matrix factorization. That reduction is good but not great. The full idea of the **FFT** is much more powerful. It saves much more than half the time.



The Full FFT by Recursion

If you have read this far, you have probably guessed what comes next. We reduced F_n to $F_{n/2}$. ***Keep going to*** $F_{n/4}$. The matrices F_{512} lead to F_{256} (in four copies). Then 256 leads to 128. *That is recursion.* It is a basic principle of many fast algorithms, and here is the second stage with four copies of $F = F_{256}$ and $D = D_{256}$:

$$\begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} = \begin{bmatrix} I & D & & \\ I & -D & & \\ & & I & D \\ & & I & -D \end{bmatrix} \begin{bmatrix} F & & & \\ & F & & \\ & & F & \\ & & & F \end{bmatrix} \begin{bmatrix} \text{pick} & 0,4,8,\cdots \\ \text{pick} & 2,6,10,\cdots \\ \text{pick} & 1,5,9,\cdots \\ \text{pick} & 3,7,11,\cdots \end{bmatrix}.$$

We will count the individual multiplications, to see how much is saved. Before the **FFT** was invented, the count was the usual $n^2 = (1024)^2$. This is about a million multiplications. I am not saying that they take a long time. The cost becomes large when we have many, many transforms to do—which is typical. Then the saving by the FFT is also large:

The final count for size $n = 2^\ell$ is reduced from n^2 to $\frac{1}{2}n\ell$.

The number 1024 is 2^{10}, so $\ell = 10$. The original count of $(1024)^2$ is reduced to (5)(1024). The saving is a factor of 200. A million is reduced to five thousand. That is why the FFT has revolutionized signal processing.

Here is the reasoning behind $\frac{1}{2}n\ell$. There are ℓ levels, going from $n = 2^\ell$ down to $n = 1$. Each level has $n/2$ multiplications from the diagonal D's, to reassemble the half-size outputs from the lower level. This yields the final count $\frac{1}{2}n\ell$, which is $\frac{1}{2}n \log_2 n$.

One last note about this remarkable algorithm. There is an amazing rule for the order that the c's enter the FFT, after all the even-odd permutations. Write the numbers 0 to $n-1$ in binary (base 2). *Reverse the order of their digits.* The complete picture shows the bit-reversed order at the start, the $\ell = \log_2 n$ steps of the recursion, and the final output $y_0, \ldots, y_{n-1}$ which is F_n times $\boldsymbol{c}$.

The book ends with that very fundamental idea, a matrix multiplying a vector.

Thank you for studying linear algebra. I hope you enjoyed it, and I very much hope you will use it. It was a pleasure to write about this tremendously useful subject.

Problem Set 10.3

1 Multiply the three matrices in equation (3) and compare with F. In which six entries do you need to know that $i^2 = -1$?

2 Invert the three factors in equation (3) to find a fast factorization of F^{-1}.

3 F is symmetric. So transpose equation (3) to find a new Fast Fourier Transform!

4 All entries in the factorization of F_6 involve powers of w_6 = sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & \\ & F_3 \end{bmatrix} \begin{bmatrix} & P & \end{bmatrix}.$$

Write down these matrices with $1, w_6, w_6^2$ in D and $w_3 = w_6^2$ in F_3. Multiply!

5 If $\boldsymbol{v} = (1,0,0,0)$ and $\boldsymbol{w} = (1,1,1,1)$, show that $F\boldsymbol{v} = \boldsymbol{w}$ and $F\boldsymbol{w} = 4\boldsymbol{v}$. Therefore $F^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $F^{-1}\boldsymbol{v} =$ _____.

6 What is F^2 and what is F^4 for the 4 by 4 Fourier matrix?

7 Put the vector $\boldsymbol{c} = (1,0,1,0)$ through the three steps of the FFT to find $\boldsymbol{y} = F\boldsymbol{c}$. Do the same for $\boldsymbol{c} = (0,1,0,1)$.

8 Compute $y = F_8 c$ by the three FFT steps for $c = (1,0,1,0,1,0,1,0)$. Repeat the computation for $c = (0,1,0,1,0,1,0,1)$.

9 If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the ____ and ____ roots of 1.

10 (a) Draw all the sixth roots of 1 on the unit circle. Prove they add to zero.

(b) What are the three cube roots of 1? Do they also add to zero?

11 The columns of the Fourier matrix F are the *eigenvectors* of the cyclic permutation P. Multiply PF to find the eigenvalues λ_1 to λ_4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}.$$

This is $PF = F\Lambda$ or $P = F\Lambda F^{-1}$. The eigenvector matrix (usually S) is F.

12 The equation $\det(P - \lambda I) = 0$ is $\lambda^4 = 1$. This shows again that the eigenvalue matrix Λ is ____. Which permutation P has eigenvalues = cube roots of 1?

13 (a) Two eigenvectors of C are $(1,1,1,1)$ and $(1, i, i^2, i^3)$. Find the eigenvalues.

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad C \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} = e_2 \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}.$$

(b) $P = F\Lambda F^{-1}$ immediately gives $P^2 = F\Lambda^2 F^{-1}$ and $P^3 = F\Lambda^3 F^{-1}$. Then $C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3 = F(c_0 I + c_1 \Lambda + c_2 \Lambda^2 + c_3 \Lambda^3) F^{-1} = FEF^{-1}$. That matrix E in parentheses is diagonal. It contains the ____ of C.

14 Find the eigenvalues of the "periodic" $-1, 2, -1$ matrix from $E = 2I - \Lambda - \Lambda^3$, with the eigenvalues of P in Λ. The -1's in the corners make this matrix periodic:

$$C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{has } c_0 = 2, c_1 = -1, c_2 = 0, c_3 = -1.$$

15 ***Fast convolution.*** To multiply C times a vector x, we can multiply $F(E(F^{-1}x))$ instead. The direct way uses n^2 separate multiplications. Knowing E and F, the second way uses only $n \log_2 n + n$ multiplications. How many of those come from E, how many from F, and how many from F^{-1}?

16 Why is row i of $\overline{F}$ the same as row $N - i$ of F (numbered 0 to $N - 1$)?

Solutions to Selected Exercises

Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

4 $3v + w = (7,5)$ and $cv + dw = (2c + d, c + 2d)$.

6 The components of every $cv + dw$ add to zero. $c = 3$ and $d = 9$ give $(3,3,-6)$.

9 The fourth corner can be $(4,4)$ or $(4,0)$ or $(-2,2)$.

11 Four more corners $(1,1,0), (1,0,1), (0,1,1), (1,1,1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Centers of faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.

12 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

13 Sum = zero vector. Sum = -2:00 vector = 8:00 vector. 2:00 is 30° from horizontal $= (\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

16 All combinations with $c + d = 1$ are on the line that passes through v and w. The point $V = -v + 2w$ is on that line but it is beyond w.

17 All vectors $cv + cw$ are on the line passing through $(0,0)$ and $u = \frac{1}{2}v + \frac{1}{2}w$. That line continues out beyond $v + w$ and back beyond $(0,0)$. With $c \geq 0$, half of this line is removed, leaving a *ray* that starts at $(0,0)$.

20 (a) $\frac{1}{3}u + \frac{1}{3}v + \frac{1}{3}w$ is the center of the triangle between u, v and w; $\frac{1}{2}u + \frac{1}{2}w$ lies between u and w (b) To fill the triangle keep $c \geq 0, d \geq 0, e \geq 0$, and $c + d + e = \mathbf{1}$.

22 The vector $\frac{1}{2}(u + v + w)$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

25 (a) For a line, choose $u = v = w$ = any nonzero vector (b) For a plane, choose u and v in different directions. A combination like $w = u + v$ is in the same plane.

Problem Set 1.2, page 19

3 Unit vectors $v/\|v\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $w/\|w\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{v}{\|v\|} \cdot \frac{w}{\|w\|} = \frac{24}{25}$. The vectors $w, u, -w$ make $0°, 90°, 180°$ angles with w.

4 (a) $v \cdot (-v) = -1$ (b) $(v + w) \cdot (v - w) = v \cdot v + w \cdot v - v \cdot w - w \cdot w = 1 + (\ \ ) - (\ \ ) - 1 = 0$ so $\theta = 90°$ (notice $v \cdot w = w \cdot v$) (c) $(v - 2w) \cdot (v + 2w) = v \cdot v - 4w \cdot w = 1 - 4 = -3$.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v}$. All vectors (x, y, z) with $x+y+z = 0$ lie on a *plane*. All vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line*.

9 If $v_2 w_2 / v_1 w_1 = -1$ then $v_2 w_2 = -v_1 w_1$ or $v_1 w_1 + v_2 w_2 = \boldsymbol{v} \cdot \boldsymbol{w} = 0$: perpendicular!

11 $\boldsymbol{v} \cdot \boldsymbol{w} < 0$ means angle $> 90°$; these $\boldsymbol{w}$'s fill half of 3-dimensional space.

12 $(1, 1)$ perpendicular to $(1, 5) - c(1, 1)$ if $6 - 2c = 0$ or $c = 3$; $\boldsymbol{v} \cdot (\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v} \cdot \boldsymbol{w} / \boldsymbol{v} \cdot \boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to perpendicular vectors.

15 $\frac{1}{2}(x + y) = (2 + 8)/2 = 5$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$.

17 $\cos\alpha = 1/\sqrt{2}, \cos\beta = 0, \cos\gamma = -1/\sqrt{2}$. For any vector $\boldsymbol{v}$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

21 $2\boldsymbol{v} \cdot \boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v} + \boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

22 $v_1^2 w_1^2 + 2 v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \le v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2 w_2^2 + v_2^2 w_1^2 - 2 v_1 w_1 v_2 w_2$ which is $(v_1 w_2 - v_2 w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta - a) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = v_1 w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2 w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. This is $\cos\theta$ because $\beta - \alpha = \theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.

28 Three vectors in the plane could make angles $> 90°$ with each other: $(1, 0), (-1, 4), (-1, -4)$. Four vectors could not do this (360° total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$?

29 Try $\boldsymbol{v} = (1, 2, -3)$ and $\boldsymbol{w} = (-3, 1, 2)$ with $\cos\theta = \frac{-7}{14}$ and $\theta = 120°$. Write $\boldsymbol{v} \cdot \boldsymbol{w} = xz + yz + xy$ as $\frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$. If $x + y + z = 0$ this is $-\frac{1}{2}(x^2 + y^2 + z^2) = -\frac{1}{2}\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -\frac{1}{2}$.

Problem Set 1.3, page 29

1 $2\boldsymbol{s}_1 + 3\boldsymbol{s}_2 + 4\boldsymbol{s}_3 = (2, 5, 9)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x} = (2, 3, 4)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row } 1) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

2 The solutions are $y_1 = 1, y_2 = 0, y_3 = 0$ (right side = column 1) and $y_1 = 1, y_2 = 3, y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

4 The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\frac{1}{2}\boldsymbol{w}_1 - \boldsymbol{w}_2 + \frac{1}{2}\boldsymbol{w}_3$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\boldsymbol{r}_2 = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual.

7 All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_3 \cdot \boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \quad &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ -x_4 &= b_5 \end{aligned}$$

Add equations 1, 3, 5
The left side of the sum is zero
The right side is $b_1 + b_3 + b_5$
There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a,b) = (3,6)$ and $(c,d) = (1,2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d are equal!

Problem Set 2.1, page 40

1 The columns are $\boldsymbol{i} = (1,0,0)$ and $\boldsymbol{j} = (0,1,0)$ and $\boldsymbol{k} = (0,0,1)$ and $\boldsymbol{b} = (2,3,4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

4 If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1,-1,2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5,1,0)$. Halfway between those is $(3,0,1)$.

6 Equation 1 + equation 2 − equation 3 is now $0 = -4$. Line misses plane; *no solution.*

8 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3,3,3,2)$ is $\boldsymbol{x} = (0,0,1,2)$ if A has columns $(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)$. The equations are $x + y + z + t = 3, y + z + t = 3, z + t = 3, t = 2$.

11 $A\boldsymbol{x}$ equals $(14,22)$ and $(0,0)$ and $(9,7)$.

14 $2x + 3y + z + 5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [2 \;\; 3 \;\; 1 \;\; 5]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane.*

16 $90°$ rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $180°$ rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

22 The dot product $Ax = [1 \ \ 4 \ \ 5] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = (1 by 3)(3 by 1) is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 $A = [1 \ \ 2 \ \ ; \ \ 3 \ \ 4]$ and $\boldsymbol{x} = [5 \ \ -2]'$ and $\boldsymbol{b} = [1 \ \ 7]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

25 **ones**$(4,4) *$ **ones**$(4,1) = [4 \ \ 4 \ \ 4 \ \ 4]'$; $B * \boldsymbol{w} = [10 \ \ 10 \ \ 10 \ \ 10]'$.

28 The row picture shows four *lines* in the 2D plane. The column picture is in *four-dimensional* space. No solution unless the right side is a combination of *the two columns*.

29 $\boldsymbol{u}_7, \boldsymbol{v}_7, \boldsymbol{w}_7$ are all close to (.6, .4). Their components still add to 1.

30 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$ = *steady state s*. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

31 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15,15,15)$; $M_4(1,1,1,1) = (34,34,34,34)$ because $1 + 2 + \cdots + 16 = 136$ which is 4(34).

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

33 $\boldsymbol{w} = (5,7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x}$ = sum of each row $= 1+\cdots+9 = 45$ for Sudoku matrices. 6 row orders (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1) are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 51

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like (8, 0) and (0, 4).

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

14 Subtract 2 times row 1 from row 2 to reach $(d-10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system becomes singular.

15 The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. A solution is $(1,1,-1)$.

17 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 = column 1, then column 2 has no pivot.

19 Row 2 becomes $3y-4z=5$, then row 3 becomes $(q+4)z=t-5$. If $q=-4$ the system is singular — no third pivot. Then if $t=5$ the third equation is $0=0$. Choosing $z=1$ the equation $3y-4z=5$ gives $y=3$ and equation 1 gives $x=-9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1+2=$ row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution.

25 $a=2$ (equal columns), $a=4$ (equal rows), $a=0$ (zero column).

28 $A(2,:)=A(2,:)-3*A(1,:)$ will subtract 3 times row 1 from row 2.

29 Pivots 2 and 3 can be arbitrarily large. I believe their averages are infinite! *With row exchanges* in MATLAB's lu code, the averages are much more stable (and should be predictable, also for randn with normal instead of uniform probability distribution).

30 If $A(5,5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1,\ldots,j$ of A. If $A\boldsymbol{x}=\mathbf{0}$ then $U\boldsymbol{x}=\mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U is the diagonal of A when A is *lower triangular.*

Problem Set 2.3, page 63

1 $E_{21}=\begin{bmatrix}1&0&0\\-5&1&0\\0&0&1\end{bmatrix}$, $E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&7&1\end{bmatrix}$, $P=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$.

3 $\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$ $M=E_{32}E_{31}E_{21}=\begin{bmatrix}1&0&0\\-4&1&0\\10&-2&1\end{bmatrix}$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot.*

9 $M=\begin{bmatrix}1&0&0\\0&0&1\\-1&1&0\end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$; $\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$; $E_{31}E_{13}=\begin{bmatrix}2&0&1\\0&1&0\\1&0&1\end{bmatrix}$. Test on the identity matrix!

12 The first product is $\begin{bmatrix}9&8&7\\6&5&4\\3&2&1\end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix}1&2&3\\0&1&-2\\0&2&-3\end{bmatrix}$.

14 E_{21} has $-\ell_{21}=\frac{1}{2}$, E_{32} has $-\ell_{32}=\frac{2}{3}$, E_{43} has $-\ell_{43}=\frac{3}{4}$. Otherwise the E's match I.

18 $EF=\begin{bmatrix}1&0&0\\a&1&0\\b&c&1\end{bmatrix}$, $FE=\begin{bmatrix}1&0&0\\a&1&0\\b+ac&c&1\end{bmatrix}$, $E^2=\begin{bmatrix}1&0&0\\2a&1&0\\2b&0&1\end{bmatrix}$, $F^3=\begin{bmatrix}1&0&0\\0&1&0\\0&3c&1\end{bmatrix}$.

22 (a) $\sum a_{3j}x_j$ (b) $a_{21}-a_{11}$ (c) $a_{21}-2a_{11}$ (d) $(E_{21}Ax)_1 = (Ax)_1 = \sum a_{1j}x_j$.

25 The last equation becomes $0 = 3$. If the original 6 is 3, then row 1 + row 2 = row 3.

27 (a) No solution if $d=0$ and $c \neq 0$ (b) Many solutions if $d=0=c$. No effect from a, b.

28 $A = AI = A(BC) = (AB)C = IC = C$. That middle equation is crucial.

30 $EM = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ then $FEM = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ then $EFEM = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ then $EEFEM = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$. So after inverting with $E^{-1} = A$ and $F^{-1} = B$ this is $M = ABAAB$.

Problem Set 2.4, page 75

2 (a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B) (d) (Row 1 of C)D(column 1 of E).

5 (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

7 (a) True (b) False (c) True (d) False.

9 $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF) = (EA)F$: Matrix multiplication is *associative*.

11 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

15 (a) mn (use every entry of A) (b) $mnp = p \times$part (a) (c) n^3 (n^2 dot products).

16 (a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

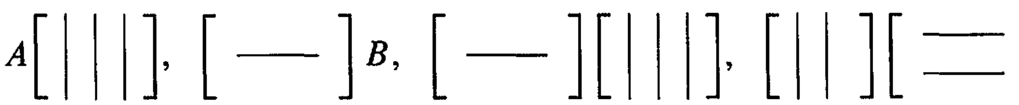
18 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

19 (a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

22 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;
$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$. You can find more examples.

24 $(A_1)^n = \begin{bmatrix} 2^n & 2^n-1 \\ 0 & 1 \end{bmatrix}$, $(A_2)^n = 2^{n-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$.

27 (a) (row 3 of A)$\cdot$(column 1 of B) and (row 3 of A)$\cdot$(column 2 of B) are both zero.
(b) $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ x \\ x \end{bmatrix}\begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$: **both upper**.

28 A times B with cuts 

30 In 29, $c = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $D - cb/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ in the lower corner of EA.

32 A times $X = [\, x_1 \;\; x_2 \;\; x_3 \,]$ will be the identity matrix $I = [\, Ax_1 \;\; Ax_2 \;\; Ax_3 \,]$.

33 $b = \begin{bmatrix}3\\5\\8\end{bmatrix}$ gives $x = 3x_1 + 5x_2 + 8x_3 = \begin{bmatrix}3\\8\\16\end{bmatrix}$; $A = \begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}$ will have those $x_1 = (1,1,1), x_2 = (0,1,1), x_3 = (0,0,1)$ as columns of its "inverse" A^{-1}.

35 $A = \begin{bmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{bmatrix}$, $A^2 = \begin{bmatrix}2&0&2&0\\0&2&0&2\\2&0&2&0\\0&2&0&2\end{bmatrix}$,

aba, ada	**cba, cda**	These show
bab, bcb	**dab, dcb**	16 2-step
abc, adc	**cbc, cdc**	paths in
bad, bcd	**dad, dcd**	the graph

Problem Set 2.5, page 89

1 $A^{-1} = \begin{bmatrix}0&\frac{1}{4}\\\frac{1}{3}&0\end{bmatrix}$ and $B^{-1} = \begin{bmatrix}\frac{1}{2}&0\\-1&\frac{1}{2}\end{bmatrix}$ and $C^{-1} = \begin{bmatrix}7&-4\\-5&3\end{bmatrix}$.

7 (a) In $Ax = (1,0,0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 (a) The vector $x = (1,1,-1)$ solves $Ax = \mathbf{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.

12 Multiply $C = AB$ on the left by A^{-1} and on the right by C^{-1}. Then $A^{-1} = BC^{-1}$.

14 $B^{-1} = A^{-1}\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1} = A^{-1}\begin{bmatrix}1&0\\-1&1\end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

16 $\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}d&-b\\-c&a\end{bmatrix} = \begin{bmatrix}ad-bc&0\\0&ad-bc\end{bmatrix}$. The inverse of each matrix is the other divided by $ad - bc$

18 $A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

21 Six of the sixteen 0 − 1 matrices are invertible, including all four with three 1's.

22 $\begin{bmatrix}1&3&1&0\\2&7&0&1\end{bmatrix} \to \begin{bmatrix}1&3&1&0\\0&1&-2&1\end{bmatrix} \to \begin{bmatrix}1&0&7&-3\\0&1&-2&1\end{bmatrix} = \begin{bmatrix}I & A^{-1}\end{bmatrix}$;

$\begin{bmatrix}1&4&1&0\\3&9&0&1\end{bmatrix} \to \begin{bmatrix}1&4&1&0\\0&-3&-3&1\end{bmatrix} \to \begin{bmatrix}1&0&-3&4/3\\0&1&1&-1/3\end{bmatrix} = \begin{bmatrix}I & A^{-1}\end{bmatrix}$.

24 $\begin{bmatrix}1&a&b&1&0&0\\0&1&c&0&1&0\\0&0&1&0&0&1\end{bmatrix} \to \begin{bmatrix}1&a&0&1&0&-b\\0&1&0&0&1&-c\\0&0&1&0&0&1\end{bmatrix} \to \begin{bmatrix}1&0&0&1&-a&ac-b\\0&1&0&0&1&-c\\0&0&1&0&0&1\end{bmatrix}$.

27 $A^{-1} = \begin{bmatrix}1&0&0\\-2&1&-3\\0&0&1\end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$.

31 Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix}a&0&-b\\-a&a&0\\0&-a&a\end{bmatrix}$.

33 $x = (1,1,\ldots,1)$ has $Px = Qx$ so $(P-Q)x = \mathbf{0}$.

34 $\begin{bmatrix}I&0\\-C&I\end{bmatrix}$ and $\begin{bmatrix}A^{-1}&0\\-D^{-1}CA^{-1}&D^{-1}\end{bmatrix}$ and $\begin{bmatrix}-D&I\\I&0\end{bmatrix}$.

35 A can be invertible with diagonal zeros. B is singular because each row adds to zero.

38 The three Pascal matrices have $P = LU = LL^{\mathrm{T}}$ and then $\mathrm{inv}(P) = \mathrm{inv}(L^{\mathrm{T}})\mathrm{inv}(L)$.

42 $MM^{-1} = (I_n - UV)\,(I_n + U(I_m - VU)^{-1}V)$ (this is testing formula **3**)
$= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V$ (keep simplifying)
$= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n$ (formulas **1, 2, 4** are similar)

43 4 by 4 still with $T_{11} = 1$ has pivots 1, 1, 1, 1; reversing to $T^* = UL$ makes $T^*_{44} = 1$.

44 Add the equations $C\boldsymbol{x} = \boldsymbol{b}$ to find $0 = b_1 + b_2 + b_3 + b_4$. Same for $F\boldsymbol{x} = \boldsymbol{b}$.

Problem Set 2.6, page 102

3 $\ell_{31} = 1$ and $\ell_{32} = 2$ (and $\ell_{33} = 1$): reverse steps to get $A\boldsymbol{u} = \boldsymbol{b}$ from $U\boldsymbol{x} = \boldsymbol{c}$: 1 times $(x+y+z = 5)$+2 times $(y+2z = 2)$+1 times $(z = 2)$ gives $x+3y+6z = 11$.

4 $Lc = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$; $\quad U\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix}\begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$; $\boldsymbol{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$.

6 $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{bmatrix}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U$. Then $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}U$ is the same as $E_{21}^{-1}E_{32}^{-1}U = LU$. The multipliers $\ell_{21}, \ell_{32} = 2$ fall into place in L.

10 $c = 2$ leads to zero in the second pivot position: exchange rows and not singular. $c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

12 $A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU$; **$U$ is L^{T}**

$\begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & -4 & \\ & & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$.

14 $\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}$. Need $\begin{matrix} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{matrix}$

15 $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\boldsymbol{c} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ gives $\boldsymbol{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Then $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ gives $\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

$A\boldsymbol{x} = \boldsymbol{b}$ is $LU\boldsymbol{x} = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$. Forward to $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boldsymbol{c}$.

18 (a) Multiply $LDU = L_1D_1U_1$ by inverses to get $L_1^{-1}LD = D_1U_1U^{-1}$. The left side is lower triangular, the right side is upper triangular $\Rightarrow$ both sides are diagonal.
(b) L, U, L_1, U_1 have diagonal 1's so $D = D_1$. Then $L_1^{-1}L$ and U_1U^{-1} are both I.

20 A tridiagonal T has 2 nonzeros in the pivot row and only one nonzero below the pivot (one operation to find ℓ and then one for the new pivot!). $T =$ bidiagonal L times bidiagonal U.

23 The 2 by 2 upper submatrix A_2 has the first two pivots 5, 9. Reason: Elimination on A starts in the upper left corner with elimination on A_2.

24 The upper left blocks all factor at the same time as A: A_k is L_kU_k.

25 The i, j entry of L^{-1} is j/i for $i \geq j$. And $L_{i\,i-1}$ is $(1-i)/i$ below the diagonal

26 $(K^{-1})_{ij} = j(n-i+1)/(n+1)$ for $i \geq j$ (and symmetric): $(n+1)K^{-1}$ looks good.

Problem Set 2.7, page 115

2 $(AB)^{\mathrm{T}}$ is not $A^{\mathrm{T}}B^{\mathrm{T}}$ *except when* $AB = BA$. Transpose that to find: $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$.

4 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. The diagonal of $A^{\mathrm{T}}A$ has dot products of columns of A with themselves. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A =$ zero matrix.

6 $M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = C$ and $D^{\mathrm{T}} = D$.

8 The 1 in row 1 has n choices; then the 1 in row 2 has $n-1$ choices ... ($n!$ overall).

10 $(3,1,2,4)$ and $(2,3,1,4)$ keep 4 in place; 6 more even P's keep 1 or 2 or 3 in place; $(2,1,4,3)$ and $(3,4,1,2)$ exchange 2 pairs. $(1,2,3,4)$, $(4,3,2,1)$ make 12 even P's.

14 The i, j entry of PAP is the $n-i+1, n-j+1$ entry of A. Diagonal will reverse order.

18 (a) $5+4+3+2+1 = 15$ independent entries if $A = A^{\mathrm{T}}$ (b) L has 10 and D has 5; total 15 in LDL^{T} (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices.

20 $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}\begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = LDL^{\mathrm{T}}.$$

22 $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$

24 $PA = LU$ is $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$. If we wait to exchange and a_{12} is the pivot, $A = L_1P_1U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

26 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

31 $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$; $A^{\mathrm{T}}y = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix}\begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$ 1 truck 1 plane

32 $Ax \cdot y$ is the *cost* of inputs while $x \cdot A^{\mathrm{T}}y$ is the *value* of outputs.

33 $P^3 = I$ so three rotations for 360°; P rotates around $(1,1,1)$ by 120°.

36 These are groups: Lower triangular with diagonal 1's, diagonal invertible D, permutations P, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

37 Certainly B^{T} is northwest. B^2 is a full matrix! B^{-1} is southeast: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. The rows of B are in reverse order from a lower triangular L, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the *columns* in reverse order from L^{-1}. So B^{-1} is *southeast*. Northwest $B = PL$ times southeast PU is $(PLP)U =$ upper triangular.

38 There are $n!$ permutation matrices of order n. Eventually *two powers of P must be the same*: If $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix} \text{ is 5 by 5 with } P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P^6 = I.$$

Problem Set 3.1, page 127

1 $x + y \ne y + x$ and $x + (y + z) \ne (x + y) + z$ and $(c_1 + c_2)x \ne c_1 x + c_2 x$.

3 (a) cx may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-x$
(b) $c(x + y)$ is the usual $(xy)^c$, while $cx + cy$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(2 + 1) = 8$. The zero vector is the number 1.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) Matrices whose main diagonal is all zero.

9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$
(b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

15 (a) Two planes through $(0, 0, 0)$ probably intersect in a line through $(0, 0, 0)$
(b) The plane and line probably intersect in the point $(0, 0, 0)$
(c) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.

20 (a) Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Solution only if $b_3 = -b_1$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.

$$[A \;\; b] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix} \begin{matrix}\text{(larger column space)} \\ \text{(no solution to } Ax = \boldsymbol{b}) \end{matrix} \quad \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix} \begin{matrix}(\boldsymbol{b} \text{ is in column space}) \\ (Ax = \boldsymbol{b} \text{ has a solution}) \end{matrix}$$

25 The solution to $Az = \boldsymbol{b} + \boldsymbol{b}^*$ is $z = x + y$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\boldsymbol{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

30 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S} + \boldsymbol{T}$, then $\boldsymbol{u} = \boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{v} = \boldsymbol{s}_2 + \boldsymbol{t}_2$. So $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$ is also in $\boldsymbol{S} + \boldsymbol{T}$. And so is $c\boldsymbol{u} = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$: *a subspace.*

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S} \cup \boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S} + \boldsymbol{T}$ is the whole plane that they span.

31 If $\boldsymbol{S} = \boldsymbol{C}(A)$ and $\boldsymbol{T} = \boldsymbol{C}(B)$ then $\boldsymbol{S} + \boldsymbol{T}$ is the column space of $M = [\,A \;\; B\,]$.

32 The columns of AB are combinations of the columns of A. So all columns of $[\,A \;\; AB\,]$ are already in $\boldsymbol{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ when A is *invertible.*

Problem Set 3.2, page 140

2 (a) Free variables x_2, x_4, x_5 and solutions $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$, $(0, 0, -3, 0, 1)$
(b) Free variable x_3: solution $(1, -1, 1)$. Special solution for each free variable.

4 $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, R has the same nullspace as U and A.

6 (a) Special solutions $(3,1,0)$ and $(5,0,1)$ (b) $(3,1,0)$. Total of pivot and free is n.

8 $R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ with $I = [\,1\,]$; $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10 (a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I, R = I$.

14 If column 1 = column 5 then x_5 is a free variable. Its special solution is $(-1,0,0,0,1)$.

16 The nullspace contains only $\boldsymbol{x} = \boldsymbol{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

20 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1,0,1,0,1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (a line in $\mathbf{R}^5$).

24 This construction is impossible: 2 pivot columns and 2 free variables, only 3 columns.

26
30 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$ and also (a)(b)(c) are all false. Notice rref$(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

32 Any zero rows come after these rows: $R = [\,1 \ \ -2 \ \ -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

33 (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) All 8 matrices are R's!

35 The nullspace of $B = [\,A \ \ A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{y} \end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

36 If $C\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x} = \boldsymbol{0}$ and $B\boldsymbol{x} = \boldsymbol{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection.*

37 *Currents*: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops.

Problem Set 3.3, page 151

1 (a) and (c) are correct; (d) is false because R might have 1's in nonpivot columns.

3 $R_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_B = \begin{bmatrix} R_A & R_A \end{bmatrix}$ $R_C \longrightarrow \begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix} \longrightarrow$ Zero rows go to the bottom

5 I think $R_1 = A_1, R_2 = A_2$ is true. But $R_1 - R_2$ may have -1's in some pivots.

7 Special solutions in $N = [\,-2 \ \ -4 \ \ 1 \ \ 0;\ -3 \ \ -5 \ \ 0 \ \ 1\,]$ and $[\,1 \ \ 0 \ \ 0;\ 0 \ \ -2 \ \ 1\,]$.

13 P has rank r (the same as A) because elimination produces the same pivot columns.

14 The rank of R^{T} is also r. The example matrix A has rank 2 with invertible S:

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

16 $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})(\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless the inner product is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

18 If we know that $\text{rank}(B^TA^T) \le \text{rank}(A^T)$, then since rank stays the same for transposes, (apologies that this fact is not yet proved), we have $\text{rank}(AB) \le \text{rank}(A)$.

20 Certainly A and B have at most rank 2. Then their product AB has at most rank 2. Since BA is 3 by 3, it cannot be I even if $AB = I$.

21 (a) A and B will both have the same nullspace and row space as the R they share.
(b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

22 $A =$ (pivot columns)(nonzero rows of R) $= \begin{bmatrix}1&0\\1&4\\1&8\end{bmatrix}\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&1&0\\1&1&0\\1&1&0\end{bmatrix} +$
$\begin{bmatrix}0&0&0\\0&0&4\\0&0&8\end{bmatrix}$. $B = \begin{bmatrix}2&2\\2&3\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix} =$ columns times rows $= \begin{bmatrix}2&0\\2&0\end{bmatrix} + \begin{bmatrix}0&2\\0&3\end{bmatrix}$

26 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

27 $R = \begin{bmatrix}I&F\\0&0\end{bmatrix} = \begin{bmatrix}r\text{ by }r & r\text{ by }n-r\\ m-r\text{ by }r & m-r\text{ by }n-r\end{bmatrix}$; $\mathbf{rref}(R^T) = \begin{bmatrix}I&0\\0&0\end{bmatrix}$; $\mathbf{rref}(R^TR) =$ same R

28 The *row-column reduced echelon form* is always $\begin{bmatrix}I&0\\0&0\end{bmatrix}$; I is r by r.

Problem Set 3.4, page 163

2 $\begin{bmatrix}2&1&3&\mathbf{b}_1\\6&3&9&\mathbf{b}_2\\4&2&6&\mathbf{b}_3\end{bmatrix} \to \begin{bmatrix}2&1&3&\mathbf{b}_1\\0&0&0&\mathbf{b}_2-3\mathbf{b}_1\\0&0&0&\mathbf{b}_3-2\mathbf{b}_1\end{bmatrix}$ Then $[\,R\;\;\boldsymbol{d}\,] = \begin{bmatrix}1&1/2&3/2&\mathbf{5}\\0&0&0&\mathbf{0}\\0&0&0&\mathbf{0}\end{bmatrix}$
$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\boldsymbol{C}(A) =$ line through $(2,6,4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

4 $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3, 1, 0, 0) + x_4(0, 0, -2, 1)$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix}5b_1 - 2b_3\\ b_3 - 2b_1\end{bmatrix} = \boldsymbol{x}_p$
(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix}5b_1 - 2b_3\\ b_3 - 2b_1\\ 0\end{bmatrix} + x_3\begin{bmatrix}-1\\-1\\1\end{bmatrix}$.

8 (a) Every $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.
(b) Need $b_3 = 2b_2$, because (row 3) $-$ 2(row 2) $= \mathbf{0}$.

12 (a) $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$ (b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}$, $A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be $\boldsymbol{x}_p$
(c) $\begin{bmatrix}3&3\\3&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}6\\6\end{bmatrix}$. Then $\begin{bmatrix}1\\1\end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix}2\\0\end{bmatrix}$ (length 2)
(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \mathbf{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \mathbf{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists.* The column space is $\mathbf{R}^3$. An example is $A = [\,I \quad F\,]$ for any 3 by 2 matrix F.

18 Rank = 2; rank = 3 unless $q = 2$ (then rank = 2). Transpose has the same rank!

25 (a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

28 $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & \mathbf{-1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$.

Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I.

30 $\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & \mathbf{-4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$; $\begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$; $\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

36 If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \mathbf{0}$). If $\boldsymbol{b}$ = column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $A = C$!

Problem Set 3.5, page 178

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column 1 = $\mathbf{0}$; if $d = 0$ then b(column 1) $-$ a(column 2) = $\mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False.* The zero vector is always in the row space.

15 The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n ($m \ge n$).

18 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.

20 One basis is $(2, 1, 0)$, $(-3, 0, 1)$. A basis for the intersection with the xy plane is $(2, 1, 0)$. The normal vector $(1, -2, 3)$ is a basis for the line perpendicular to the plane.

22 (a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in $\mathbf{S}$.

25 Rank 2 if $c = 0$ and $d = 2$; rank 2 except when $c = d$ or $c = -d$.

28 $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

32 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

34 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

37 The subspace of matrices that have $AS = SA$ has dimension *three.*

39 If the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A. If $[\,A \;\; \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent, $\boldsymbol{b}$ *is* a combination of those columns. In this case $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

41 $I = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$. The six P's are dependent.

42 The dimension of S is (a) zero when $\boldsymbol{x} = \boldsymbol{0}$ (b) one when $\boldsymbol{x} = (1, 1, 1, 1)$ (c) three when $\boldsymbol{x} = (1, 1, -1, -1)$ because all rearrangements have $x_1 + \cdots + x_4 = 0$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** $\dim \mathbf{S} = 2$.

43 The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\boldsymbol{0}$. **To be proved**: All coefficients = zero.

Key idea: The part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$, so the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now $\boldsymbol{x} - \boldsymbol{x} = \boldsymbol{0}$ uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \boldsymbol{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

44 The inputs to an m by n matrix fill $\mathbf{R}^n$. The outputs (column space!) have dimension r. The nullspace has $n - r$ special solutions. The formula becomes $r + (n - r) = n$.

Problem Set 3.6, page 190

1 (a) Row and column space dimensions = 5, nullspace dimension = 4, $\dim(N(A^{\mathrm{T}}))$ $= 2$ sum $= 16 = m + n$ (b) Column space is $\mathbf{R}^3$; left nullspace contains only $\boldsymbol{0}$.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r + (n - r)$ must be 3 (c) $[\,1 \;\; 1\,]$ (d) $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

(e) *Impossible* Row space = column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $N(A)$ and $N(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

6 A: dim **2, 2, 2, 1**: Rows $(0, 3, 3, 3)$ and $(0, 1, 0, 1)$; columns $(3, 0, 1)$ and $(3, 0, 0)$; nullspace $(1, 0, 0, 0)$ and $(0, -1, 0, 1)$; $N(A^{\mathrm{T}})$ $(0, 1, 0)$. B: dim **1, 1, 0, 2** Row space (1), column space $(1, 4, 5)$, nullspace: empty basis, $N(A^{\mathrm{T}})$ $(-4, 1, 0)$ and $(-5, 0, 1)$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same (b) Same column space and left nullspace. Same rank (dimension of column space).

11 (a) No solution means that $r < m$. Always $r \leq n$. Can't compare m and n (b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \boldsymbol{0}$ is in both $N(A)$ and $C(A^{\mathrm{T}})$.

16 If $A\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

18 Row 3−2 row 2+ row 1 = zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

20 (a) Special solutions $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ are perpendicular to the rows of R (and then ER). (b) $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 1 independent solution = last row of E^{-1}. ($E^{-1}A = R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y} = \mathbf{0}$.

26 The rows of $C = AB$ are combinations of the rows of B. So rank $C \le$ rank B. Also rank $C \le$ rank A, because the columns of C are combinations of the columns of A.

29 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1.$

30 The subspaces for $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B = cA$ with $c \neq 0$.

31 (a) $AX = 0$ if each column of X is a multiple of $(1,1,1)$; dim(nullspace) = 3. (b) If $AX = B$ then all columns of B add to zero; dimension of the B's = 6. (c) $3 + 6 = \dim(M^{3\times 3}) = 9$ entries in a 3 by 3 matrix.

32 The key is equal row spaces. First row of A = combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.

Problem Set 4.1, page 202

1 Both nullspace vectors are orthogonal to the row space vector in $\mathbf{R}^3$. The column space is perpendicular to the nullspace of A^{T} (two lines in $\mathbf{R}^2$ because rank = 1).

3 (a) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ (b) Impossible, $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $C(A)$ and $N(A^{\mathrm{T}})$ is impossible: not perpendicular (d) Need $A^2 = 0$; take $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ (e) $(1,1,1)$ in the nullspace (columns add to $\mathbf{0}$) and also row space; no such matrix.

6 Multiply the equations by $y_1, y_2, y_3 = 1, 1, -1$. Equations add to $0 = 1$ so no solution: $\boldsymbol{y} = (1,1,-1)$ is in the left nullspace. $A\boldsymbol{x} = \boldsymbol{b}$ would need $0 = (\boldsymbol{y}^{\mathrm{T}}A)\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1$.

8 $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space and $\boldsymbol{x}_n$ is in the nullspace. Then $A\boldsymbol{x}_n = \mathbf{0}$ and $A\boldsymbol{x} = A\boldsymbol{x}_r + A\boldsymbol{x}_n = A\boldsymbol{x}_r$. All $A\boldsymbol{x}$ are in $C(A)$.

9 $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ is also in the nullspace of A^{T}. So $A\boldsymbol{x}$ is perpendicular to itself. Conclusion: $A\boldsymbol{x} = \mathbf{0}$ if $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$.

10 (a) With $A^{\mathrm{T}} = A$, the column and row spaces are the same (b) $\boldsymbol{x}$ is in the nullspace and $\boldsymbol{z}$ is in the column space = row space: so these "eigenvectors" have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z} = 0$.

12 $\boldsymbol{x}$ splits into $\boldsymbol{x}_r + \boldsymbol{x}_n = (1,-1) + (1,1) = (2,0)$. Notice $N(A^{\mathrm{T}})$ is a plane $(1,0) = (1,1)/2 + (1,-1)/2 = \boldsymbol{x}_r + \boldsymbol{x}_n$.

13 $V^{\mathrm{T}}W$ = zero makes each basis vector for $\boldsymbol{V}$ orthogonal to each basis vector for $\boldsymbol{W}$. Then every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$ (combinations of the basis vectors).

14 $Ax = B\widehat{x}$ means that $\begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} x \\ -\widehat{x} \end{bmatrix} = \mathbf{0}$. Three homogeneous equations in four unknowns always have a nonzero solution. Here $x = (3,1)$ and $\widehat{x} = (1,0)$ and $Ax = B\widehat{x} = (5,6,5)$ is in both column spaces. Two planes in $\mathbf{R}^3$ must share a line.

16 $A^{\mathrm{T}}y = \mathbf{0}$ leads to $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}A^{\mathrm{T}}y = 0$. Then $y \perp Ax$ and $N(A^{\mathrm{T}}) \perp C(A)$.

18 $S^{\perp}$ is the nullspace of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Therefore $S^{\perp}$ is a *subspace* even if S is not.

21 For example $(-5,0,1,1)$ and $(0,1,-1,0)$ span $S^{\perp}$=nullspace of $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$.

23 x in $V^{\perp}$ is perpendicular to any vector in V. Since V contains all the vectors in S, x is also perpendicular to any vector in S. So every x in $V^{\perp}$ is also in $S^{\perp}$.

28 (a) $(1,-1,0)$ is in both planes. Normal vectors are perpendicular, but planes still intersect! (b) Need *three* orthogonal vectors to span the whole orthogonal complement. (c) Lines can meet at the zero vector without being orthogonal.

30 When $AB = 0$, the column space of B is contained in the nullspace of A. Therefore the dimension of $C(B) \le$ dimension of $N(A)$. This means $\text{rank}(B) \le 4 - \text{rank}(A)$.

31 null(N') produces a basis for the *row space* of A (perpendicular to $N(A)$).

32 We need $r^{\mathrm{T}}n = 0$ and $c^{\mathrm{T}}\ell = 0$. All possible examples have the form acr^{T} with $a \neq 0$.

33 Both r's orthogonal to both n's, both c's orthogonal to both ℓ's, each pair independent. All A's with these subspaces have the form $[c_1\ c_2]M[r_1\ r_2]^{\mathrm{T}}$ for a 2 by 2 invertible M.

Problem Set 4.2, page 214

1 (a) $a^{\mathrm{T}}b/a^{\mathrm{T}}a = 5/3$; $p = 5a/3$; $e = (-2,1,1)/3$ (b) $a^{\mathrm{T}}b/a^{\mathrm{T}}a = -1$; $p = a$; $e = \mathbf{0}$.

3 $P_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P_1 b = \frac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$. $P_2 = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ and $P_2 b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

6 $p_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$ and $p_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$ and $p_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$. So $p_1 + p_2 + p_3 = b$.

9 Since A is invertible, $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$: project on all of $\mathbf{R}^2$.

11 (a) $p = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b = (2,3,0)$, $e = (0,0,4)$, $A^{\mathrm{T}}e = \mathbf{0}$ (b) $p = (4,4,6)$, $e = \mathbf{0}$.

15 $2A$ has the same column space as A. $\widehat{x}$ for $2A$ is *half* of $\widehat{x}$ for A.

16 $\frac{1}{2}(1,2,-1) + \frac{3}{2}(1,0,1) = (2,1,1)$. So b is in the plane. Projection shows $Pb = b$.

18 (a) $I - P$ is the projection matrix onto $(1,-1)$ in the perpendicular direction to $(1,1)$ (b) $I - P$ projects onto the plane $x + y + z = 0$ perpendicular to $(1,1,1)$.

20 $e = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $Q = \frac{ee^{\mathrm{T}}}{e^{\mathrm{T}}e} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

21 $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. So $P^2 = P$. Pb is in the column space (where P projects). Then its projection $P(Pb)$ is Pb.

24 The nullspace of A^T is *orthogonal* to the column space $C(A)$. So if $A^T\boldsymbol{b} = \mathbf{0}$, the projection of $\boldsymbol{b}$ onto $C(A)$ should be $\boldsymbol{p} = \mathbf{0}$. Check $P\boldsymbol{b} = A(A^TA)^{-1}A^T\boldsymbol{b} = A(A^TA)^{-1}\mathbf{0}$.

28 $P^2 = P = P^T$ give $P^TP = P$. Then the $(2,2)$ entry of P equals the $(2,2)$ entry of P^TP which is the length squared of column 2.

29 $A = B^T$ has independent columns, so A^TA (which is BB^T) must be invertible.

30 (a) The column space is the line through $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \dfrac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T\boldsymbol{a}} = \dfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$.

(b) The row space is the line through $\boldsymbol{v} = (1,2,2)$ and $P_R = \boldsymbol{v}\boldsymbol{v}^T/\boldsymbol{v}^T\boldsymbol{v}$. Always $P_CA = A$ (columns of A project to themselves) and $AP_R = A$. Then $P_CAP_R = A$!

31 The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ must be perpendicular to all the $\boldsymbol{a}$'s.

32 Since $P_1\boldsymbol{b}$ is in $C(A)$, $P_2(P_1\boldsymbol{b})$ equals $P_1\boldsymbol{b}$. So $P_2P_1 = P_1 = \boldsymbol{a}\boldsymbol{a}^T/\boldsymbol{a}^T\boldsymbol{a}$ where $\boldsymbol{a} = (1,2,0)$.

33 If $P_1P_2 = P_2P_1$ then $\boldsymbol{S}$ is contained in $\boldsymbol{T}$ or $\boldsymbol{T}$ is contained in $\boldsymbol{S}$.

34 BB^T is invertible as in Problem 29. Then $(A^TA)(BB^T)$ = product of r by r invertible matrices, so rank r. AB can't have rank $< r$, since A^T and B^T cannot increase the rank. *Conclusion*: A (m by r of rank r) times B (r by n of rank r) produces AB of rank r.

Problem Set 4.3, page 226

1 $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^TA = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^T\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$ gives $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$, $E = \|\boldsymbol{e}\|^2 = \mathbf{44}$

5 $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$. $A^T = [1 \ \ 1 \ \ 1 \ \ 1]$ and $A^TA = [4]$. $A^T\boldsymbol{b} = [36]$ and $(A^TA)^{-1}A^T\boldsymbol{b} = 9$ = best height C. Errors $\boldsymbol{e} = (-9,-1,-1,11)$.

7 $A = [0 \ \ 1 \ \ 3 \ \ 4]^T$, $A^TA = [26]$ and $A^T\boldsymbol{b} = [112]$. Best $D = 112/26 = 56/13$.

8 $\widehat{\boldsymbol{x}} = 56/13$, $\boldsymbol{p} = (56/13)(0,1,3,4)$. $(C,D) = (9, 56/13)$ don't match $(C,D) = (1,4)$. Columns of A were not perpendicular so we can't project separately to find C and D.

9 Parabola Project $\boldsymbol{b}$ 4D to 3D $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. $A^TA\widehat{\boldsymbol{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$.

11 (a) The best line $x = 1 + 4t$ gives the center point $\widehat{b} = 9$ when $\widehat{t} = 2$.

(b) The first equation $Cm + D\sum t_i = \sum b_i$ divided by m gives $C + D\widehat{t} = \widehat{b}$.

13 $(A^TA)^{-1}A^T(\boldsymbol{b} - A\boldsymbol{x}) = \widehat{\boldsymbol{x}} - \boldsymbol{x}$. When $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ averages to $\mathbf{0}$, so does $\widehat{\boldsymbol{x}} - \boldsymbol{x}$.

14 The matrix $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^T$ is $(A^TA)^{-1}A^T(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^TA(A^TA)^{-1}$. When the average of $(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^T$ is $\sigma^2 I$, the average of $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^T$ will be the *output covariance matrix* $(A^TA)^{-1}A^T\sigma^2A(A^TA)^{-1}$ which simplifies to $\sigma^2(A^TA)^{-1}$.

16 $\frac{1}{10}b_{10} + \frac{9}{10}\widehat{x}_9 = \frac{1}{10}(b_1 + \cdots + b_{10})$. Knowing $\widehat{x}_9$ avoids adding all b's.

18 $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \boldsymbol{0}$.

21 $\boldsymbol{e}$ is in $N(A^{\mathrm{T}})$; $\boldsymbol{p}$ is in $C(A)$; $\widehat{\boldsymbol{x}}$ is in $C(A^{\mathrm{T}})$; $N(A) = \{\boldsymbol{0}\}$ = zero vector only.

23 The square of the distance between points on two lines is $E = (y-x)^2 + (3y-x)^2 + (1+x)^2$. Derivatives $\frac{1}{2}\partial E/\partial x = 3x - 4y + 1 = 0$ and $\frac{1}{2}\partial E/\partial y = -4x + 10y = 0$. The solution is $x = -5/7, y = -2/7$; $E = 2/7$, and the minimum distance is $\sqrt{2/7}$.

25 3 points on a line: *Equal slopes* $(b_2-b_1)/(t_2-t_1) = (b_3-b_2)/(t_3-t_2)$. Linear algebra: Orthogonal to $(1,1,1)$ and (t_1,t_2,t_3) is $\boldsymbol{y} = (t_2-t_3, t_3-t_1, t_1-t_2)$ in the left nullspace. $\boldsymbol{b}$ is in the column space. Then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ is the same equal slopes condition written as $(b_2-b_1)(t_3-t_2) = (b_3-b_2)(t_2-t_1)$.

27 The shortest link connecting two lines in space is *perpendicular to those lines.*

28 Only 1 plane contains $\boldsymbol{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ unless $\boldsymbol{a}_1, \boldsymbol{a}_2$ are *dependent.* Same test for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

Problem Set 4.4, page 239

3 (a) $A^{\mathrm{T}}A$ will be $16I$ (b) $A^{\mathrm{T}}A$ will be diagonal with entries 1, 4, 9.

6 Q_1Q_2 is orthogonal because $(Q_1Q_2)^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_1^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_2 = I$.

8 If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are *orthonormal* vectors in $\mathbf{R}^5$ then $(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_1 + (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_2$ is closest to $\boldsymbol{b}$.

11 (a) Two *orthonormal* vectors are $\boldsymbol{q}_1 = \frac{1}{10}(1,3,4,5,7)$ and $\boldsymbol{q}_2 = \frac{1}{10}(-7,3,4,-5,1)$
(b) Closest in the plane: *project* $QQ^{\mathrm{T}}(1,0,0,0,0) = (0.5, -0.18, -0.24, 0.4, 0)$.

13 The multiple to subtract is $\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$. Then $\boldsymbol{B} = \boldsymbol{b} - \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = (4,0) - 2 \cdot (1,1) = (2,-2)$.

14 $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} \|\boldsymbol{a}\| & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b} \\ 0 & \|\boldsymbol{B}\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = QR.$

15 (a) $\boldsymbol{q}_1 = \frac{1}{3}(1,2,-2)$, $\boldsymbol{q}_2 = \frac{1}{3}(2,1,2)$, $\boldsymbol{q}_3 = \frac{1}{3}(2,-2,-1)$ (b) The nullspace of A^{T} contains $\boldsymbol{q}_3$ (c) $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(1,2,7) = (1,2)$.

16 The projection $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a} = 14\boldsymbol{a}/49 = 2\boldsymbol{a}/7$ is closest to $\boldsymbol{b}$; $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\| = \boldsymbol{a}/7$ is $(4,5,2,2)/7$. $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (-1,4,-4,-4)/7$ has $\|\boldsymbol{B}\| = 1$ so $\boldsymbol{q}_2 = \boldsymbol{B}$.

18 $A = \boldsymbol{a} = (1,-1,0,0)$; $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (\frac{1}{2}, \frac{1}{2}, -1, 0)$; $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{p}_A - \boldsymbol{p}_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Notice the pattern in those orthogonal $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. In $\mathbf{R}^5$, $\boldsymbol{D}$ would be $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$.

20 (a) *True* (b) *True.* $Q\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2$. $\|Q\boldsymbol{x}\|^2 = x_1^2 + x_2^2$ because $\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 = 0$.

21 The orthonormal vectors are $\boldsymbol{q}_1 = (1,1,1,1)/2$ and $\boldsymbol{q}_2 = (-5,-1,1,5)/\sqrt{52}$. Then $\boldsymbol{b} = (-4,-3,3,0)$ projects to $\boldsymbol{p} = (-7,-3,-1,3)/2$. And $\boldsymbol{b} - \boldsymbol{p} = (-1,-3,7,-3)/2$ is orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

22 $A = (1,1,2)$, $B = (1,-1,0)$, $C = (-1,-1,1)$. These are not yet unit vectors.

26 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \frac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \frac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

28 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

30 The wavelet matrix W has orthonormal columns. Notice $W^{-1} = W^{\mathrm{T}}$ in Section 7.3.

32 $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ across plane $y + z = 0$.

33 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal and zeros elsewhere.

Problem Set 5.1, page 251

1 $\det(2A) = 8$; $\det(-A) = (-1)^4 \det A = \frac{1}{2}$; $\det(A^2) = \frac{1}{4}$; $\det(A^{-1}) = 2 = \det(A^{\mathrm{T}})^{-1}$.

5 $|J_5| = 1$, $|J_6| = -1$, $|J_7| = -1$. Determinants $1, 1, -1, -1$ repeat so $|J_{101}| = 1$.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so det can't blow up.

10 If the entries in every row add to zero, then $(1, 1, \ldots, 1)$ is in the nullspace: singular A has det $= 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$ and *not* $-\det DC$. If n is even we can have an invertible CD.

14 $\det(A) = 36$ and the 4 by 4 second difference matrix has det $= 5$.

15 The first determinant is 0, the second is $1 - 2t^2 + t^4 = (1 - t^2)^2$.

17 Any 3 by 3 skew-symmetric K has $\det(K^{\mathrm{T}}) = \det(-K) = (-1)^3 \det(K)$. This is $-\det(K)$. But always $\det(K^{\mathrm{T}}) = \det(K)$, so we must have $\det(K) = 0$ for 3 by 3.

21 Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

23 $\det(A) = 10$, $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$, $\det(A^2) = 100$, $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$ has det $\frac{1}{10}$.
$\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = \mathbf{2}$ or $\lambda = \mathbf{5}$; those are eigenvalues.

27 $\det A = abc$, $\det B = -abcd$, $\det C = a(b - a)(c - b)$ by doing elimination.

32 Typical determinants of rand(n) are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. randn(n) with normal distribution gives $10^{31}, 10^{78}, 10^{186}$, Inf which means $\geq 2^{1024}$. MATLAB allows $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives Inf!

Problem Set 5.2, page 263

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = -1$, independent.

4 $a_{11}a_{23}a_{32}a_{44}$ gives -1, because $2 \leftrightarrow 3$, $a_{14}a_{23}a_{32}a_{41}$ gives $+1$, $\det A = 1 - 1 = 0$; $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$.

6 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms are sure zeros (b) 15 terms must be zero.

8 Some term $a_{1\alpha}a_{2\beta} \cdots a_{n\omega}$ in the big formula is not zero! Move rows $1, 2, \ldots, n$ into rows $\alpha, \beta, \ldots, \omega$. Then these nonzero a's will be on the main diagonal.

9 To get $+1$ for the even permutations the matrix needs an *even* number of -1's. For the odd P's the matrix needs an *odd* number of -1's. So six 1's and det $= 6$ are impossible five 1's and one -1 will give $AC = (ad - bc)I = (\det A)I$ max(det) $= 4$.

11 $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$. Puzzle: $\det D = 441 = (-21)^2$. *Why?*

12 $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^{\mathrm{T}} = C^{\mathrm{T}}/\det A$.

13 (a) $C_1 = 0,\ C_2 = -1,\ C_3 = 0,\ C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$.

15 The 1, 1 cofactor of the n by n matrix is E_{n-1}. The 1, 2 cofactor has a single 1 in its first column, with cofactor E_{n-2}: sign gives $-E_{n-2}$. So $E_n = E_{n-1} - E_{n-2}$. Then E_1 to E_6 is $1, 0, -1, -1, 0, 1$ and this cycle of six will repeat: $E_{100} = E_4 = -1$.

16 The 1, 1 cofactor of the n by n matrix is F_{n-1}. The 1, 2 cofactor has a 1 in column 1, with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the 1, 2 entry to find $F_n = F_{n-1} + F_{n-2}$ (so these determinants are Fibonacci numbers).

19 Since x, x^2, x^3 are all in the same row, they are never multiplied in $\det V_4$. The determinant is zero at $x = a$ or b or c, so $\det V$ has factors $(x-a)(x-b)(x-c)$. Multiply by the cofactor V_3. The Vandermonde matrix $V_{ij} = (x_i)^{j-1}$ is for fitting a polynomial $p(x) = \boldsymbol{b}$ at the points x_i. It has $\det V =$ product of all $x_k - x_m$ for $k > m$.

20 $G_2 = -1, G_3 = 2, G_4 = -3$, and $G_n = (-1)^{n-1}(n-1) =$ (product of the λ's).

24 (a) All L's have det $= 1$; $\det U_k = \det A_k = 2, 6, -6$ (b) Pivots $5, 6/5, 7/6$.

25 Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

27 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. Derivative with respect to $a_{11} =$ cofactor C_{11}.

29 There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+\,(1,1)(2,2)(3,3)(4,4)\ +\ (1,2)(2,1)(3,4)(4,3)\ -\ (1,2)(2,1)(3,3)(4,4)\ -\ (1,1)(2,2)(3,4)(4,3)\ -\ (1,1)(2,3)(3,2)(4,4)$. Total -1.

32 The problem is to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}.$$

33 The difference from 20 to 19 multiplies its 3 by 3 cofactor $= 1$: then det drops by 1.

34 (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one of those choices will be zero.

Problem Set 5.3, page 278

2 (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c/(ad - bc)$ (b) $y = \det B_2 / \det A = (fg - id)/D$.

3 (a) $x_1 = 3/0$ and $x_2 = -2/0$: *no solution* (b) $x_1 = x_2 = \mathbf{0/0}$: *undetermined.*

4 (a) $x_1 = \det([\,\boldsymbol{b}\ \ \boldsymbol{a}_2\ \ \boldsymbol{a}_3\,])/\det A$, if $\det A \neq 0$ (b) The determinant is linear in its first column so $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_2|\boldsymbol{a}_2\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_3|\boldsymbol{a}_3\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$ which is $x_1 \det A$.

6 (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$ (b) $\frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. An invertible symmetric matrix has a symmetric inverse.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. This is $(\det A)I$ and $\det A = 3$. The $1, 3$ cofactor of A is 0. Multiplying by 4 or 100: no change.

9 If we know the cofactors and $\det A = 1$, then $C^{\mathrm{T}} = A^{-1}$ and also $\det A^{-1} = 1$. Now A is the inverse of C^{T}, so A can be found from the cofactor matrix for C.

11 The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1}.

15 For $n = 5$, C contains *25* cofactors and each 4 by 4 cofactor has *24* terms. Each term needs *3* multiplications: total 1800 multiplications vs.125 for Gauss-Jordan.

17 Volume $= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$. Area of faces length of cross product $= \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \begin{matrix} -2i - 2j + 8k \\ \text{length} = 6\sqrt{2} \end{matrix}$

18 (a) Area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$ (b) $5 +$ new triangle area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

21 The maximum volume is $L_1 L_2 L_3 L_4$ reached when the edges are orthogonal in $\mathbf{R}^4$. With entries 1 and -1 all lengths are $\sqrt{4} = 2$. The maximum determinant is $2^4 = 16$, achieved in Problem 20. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved.

23 $A^{\mathrm{T}}A = \begin{bmatrix} a^{\mathrm{T}} \\ b^{\mathrm{T}} \\ c^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a^{\mathrm{T}}a & 0 & 0 \\ 0 & b^{\mathrm{T}}b & 0 \\ 0 & 0 & c^{\mathrm{T}}c \end{bmatrix}$ has $\begin{matrix} \det A^{\mathrm{T}}A & = & (\|a\|\,\|b\|\,\|c\|)^2 \\ \det A & = & \pm\|a\|\,\|b\|\,\|c\| \end{matrix}$

25 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients from $(2 + x)^n$ in Worked Example **2.4A**. Cube from $2I$ has volume 2^n.

26 The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$ (and $\frac{1}{n!}$ in $\mathbf{R}^n$)

31 Base area 10, height 2, volume 20.

35 $S = (2, 1, -1)$, area $\|PQ \times PS\| = \|(-2, -2, -1)\| = 3$. The other four corners can be $(0, 0, 0)$, $(0, 0, 2)$, $(1, 2, 2)$, $(1, 1, 0)$. The volume of the tilted box is $|\det| = 1$.

39 $AC^{\mathrm{T}} = (\det A)I$ gives $(\det A)(\det C) = (\det A)^n$. Then $\det A = (\det C)^{1/3}$ with $n = 4$. With $\det A^{-1}$ is $1/\det A$, construct A^{-1} using the cofactors. *Invert to find A.*

Problem Set 6.1, page 293

1 The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^{∞}. Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (the trace is now $0.2 + 0.3$). Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $x_1 = (1, 1)$ and $x_2 = (2, -1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved in section 6.6, Problems 18-19).

8 (a) Multiply Ax to see λx which reveals λ (b) Solve $(A - \lambda I)x = \mathbf{0}$ to find x.

10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $x_1 = (1,2)$ and $x_2 = (1,-1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 Columns of $A - \lambda_1 I$ are in the nullspace of $A - \lambda_2 I$ because $M = (A-\lambda_2 I)(A-\lambda_1 I)$ = zero matrix [this is the *Cayley-Hamilton Theorem* in Problem 6.2.32]. Notice that M has *zero eigenvalues* $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_1) = 0$ and $(\lambda_2 - \lambda_2)(\lambda_2 - \lambda_1) = 0$.

13 (a) $Pu = (uu^{\mathrm{T}})u = u(u^{\mathrm{T}}u) = u$ so $\lambda = 1$ (b) $Pv = (uu^{\mathrm{T}})v = u(u^{\mathrm{T}}v) = \mathbf{0}$
(c) $x_1 = (-1,1,0,0)$, $x_2 = (-3,0,1,0)$, $x_3 = (-5,0,0,1)$ all have $Px = 0x = \mathbf{0}$.

15 The other two eigenvalues are $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$; the three eigenvalues are $1, 1, -1$.

16 Set $\lambda = 0$ in $\det(A - \lambda I) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda)$ to find $\det A = (\lambda_1)(\lambda_2)\cdots(\lambda_n)$.

17 $\lambda_1 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc})$ and $\lambda_2 = \frac{1}{2}(a + d - \sqrt{})$ add to $a + d$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$.

19 (a) rank = 2 (b) $\det(B^{\mathrm{T}}B) = 0$ (d) eigenvalues of $(B^2 + I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{5}$.

20 Last rows are $-28, 11$ (check trace and det) and $6, -11, 6$ (to match $\det(C - \lambda I)$).

22 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues = trace = $\frac{1}{2}$).

23 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always A^2 is the zero matrix if $\lambda = 0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.32.

28 B has $\lambda = -1, -1, -1, 3$ and C has $\lambda = 1, 1, 1, -3$. Both have det $= -3$.

32 (a) u is a basis for the nullspace, v and w give a basis for the column space
(b) $x = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any cu from the nullspace
(c) If $Ax = u$ had a solution, u would be in the column space: wrong dimension 3.

34 $\det(P - \lambda I) = 0$ gives the equation $\lambda^4 = 1$. This reflects the fact that $P^4 = I$. The solutions of $\lambda^4 = 1$ are $\lambda = 1, i, -1, -i$. The real eigenvector $x_1 = (1,1,1,1)$ is not changed by the permutation P. Three more eigenvectors are (i, i^2, i^3, i^4) and $(1,-1,1,-1)$ and $(-i, (-i)^2, (-i)^3, (-i)^4)$.

36 $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$ give $\det \lambda_1\lambda_2 = 1$ and trace $\lambda_1 + \lambda_2 = -1$. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \dfrac{2\pi}{3}$ has this trace and det. So does every $M^{-1}AM$!

Problem Set 6.2, page 307

1 $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

3 If $A = S\Lambda S^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still S. $A + 2I = S(\Lambda + 2I)S^{-1} = S\Lambda S^{-1} + S(2I)S^{-1} = A + 2I$.

4 (a) False: don't know λ's (b) True (c) True (d) False: need eigenvectors of S

6 The columns of S are nonzero multiples of $(2,1)$ and $(0,1)$: either order. Same for A^{-1}.

8 $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$. $S\Lambda^k S^{-1} =$
$\dfrac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \textit{2nd component} \text{ is } F_k \\ (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2) \end{bmatrix}$.

9 (a) $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$ with $x_1 = (1, 1)$, $x_2 = (1, -2)$

(b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

12 (a) False: don't know λ (b) True: an eigenvector is missing (c) True.

13 $A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $x = (c, -c)$.

15 $A^k = S\Lambda^k S^{-1}$ approaches zero **if and only if every $|\lambda| < 1$**; $A_1^k \to A_1^\infty$, $A_2^k \to 0$.

17 $\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$, $S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $A_2^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A_2^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$,
$A_2^{10}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ is the sum of $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

19 $B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

21 trace $ST = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace TS. Diagonalizable case: the trace of $S\Lambda S^{-1}$ = trace of $(\Lambda S^{-1})S = \Lambda$: *sum of the* λ's.

24 The A's form a subspace since cA and $A_1 + A_2$ all have the same S. When $S = I$ the A's with those eigenvectors give the subspace of diagonal matrices. Dimension 4.

26 Two problems: The nullspace and column space can overlap, so x could be in both. There may not be r independent eigenvectors in the column space.

27 $R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$. $\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace is not real.
Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ can have $\sqrt{-1} = i$ and $-i$, trace 0, real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

28 $A^T = A$ gives $x^T ABx = (Ax)^T(Bx) \le \|Ax\|\|Bx\|$ by the Schwarz inequality. $B^T = -B$ gives $-x^T BAx = (Bx)^T(Ax) \le \|Ax\|\|Bx\|$. Add to get Heisenberg's Uncertainty Principle when $AB - BA = I$. Position-momentum, also time-energy.

32 If $A = S\Lambda S^{-1}$ then $(A - \lambda_1 I)\cdots(A - \lambda_n I)$ equals $S(\Lambda - \lambda_1 I)\cdots(\Lambda - \lambda_n I)S^{-1}$. The factor $\Lambda - \lambda_j I$ is zero in row j. *The product is zero in all rows* = ***zero matrix***.

33 $\lambda = 2, -1, 0$ are in Λ and the eigenvectors are in S (below). $A^k = S\Lambda^k S^{-1}$ is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\Lambda^k \frac{1}{6}\begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & -2 \\ 0 & 3 & -3 \end{bmatrix} = \frac{2^k}{6}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \frac{(-1)^k}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Check $k = 4$. The $(2, 2)$ entry of A^4 is $2^4/6 + (-1)^4/3 = 18/6 = 3$. The 4-step paths that begin and end at node 2 are 2 to 1 to 1 to 1 to 2, 2 to 1 to 2 to 1 to 2, and 2 to 1 to 3 to 1 to 2. Much harder to find the eleven 4-step paths that start and end at node 1.

35 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1 and $B^4 = I$. C has $\lambda = (1 \pm \sqrt{3}i)/2$. This is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ and $C^{1024} = -C$.

37 Columns of S times rows of ΛS^{-1} will give r rank-1 matrices ($r =$ rank of A).

Problem Set 6.3, page 325

1 $u_1 = e^{4t}\begin{bmatrix}1\\0\end{bmatrix}$, $u_2 = e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$. If $u(0) = (5,-2)$, then $u(t) = 3e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + 2e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$.

4 $d(v+w)/dt = (w-v)+(v-w) = \mathbf{0}$, so the total $v+w$ is constant. $A = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$ has $\begin{matrix}\lambda_1 = 0\\ \lambda_2 = -2\end{matrix}$ with $x_1 = \begin{bmatrix}1\\1\end{bmatrix}, x_2 = \begin{bmatrix}1\\-1\end{bmatrix}$; $\begin{matrix}v(1) = 20 + 10e^{-2}\\ w(1) = 20 - 10e^{-2}\end{matrix}$ $\begin{matrix}v(\infty) = 20\\ w(\infty) = 20\end{matrix}$

8 $\begin{bmatrix}6 & -2\\ 2 & 1\end{bmatrix}$ has $\lambda_1 = 5$, $x_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\lambda_2 = 2$, $x_2 = \begin{bmatrix}1\\2\end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches 20/10; e^{5t} dominates.

12 $A = \begin{bmatrix}0 & 1\\ -9 & 6\end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector (1, 3).

14 When A is skew-symmetric, $\|u(t)\| = \|e^{At}u(0)\|$ is $\|u(0)\|$. So e^{At} is *orthogonal*.

15 $u_p = 4$ and $u(t) = ce^t + 4$; $u_p = \begin{bmatrix}4\\2\end{bmatrix}$ and $u(t) = c_1e^t\begin{bmatrix}1\\t\end{bmatrix} + c_2e^t\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}4\\2\end{bmatrix}$.

16 Substituting $u = e^{ct}v$ gives $ce^{ct}v = Ae^{ct}v - e^{ct}b$ or $(A - cI)v = b$ or $v = (A - cI)^{-1}b =$ particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

20 The solution at time $t + T$ is also $e^{A(t+T)}u(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

21 $\begin{bmatrix}1 & 4\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}\mathbf{1} & 0\\ 0 & \mathbf{0}\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}$; $\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}e^t & 0\\ 0 & \mathbf{1}\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix} = \begin{bmatrix}e^t & 4e^t - 4\\ 0 & 1\end{bmatrix}$.

22 $A^2 = A$ gives $e^{At} = I + At + \frac{1}{2}At^2 + \cdots = I + (e^t - 1)A = \begin{bmatrix}e^t & e^t - 1\\ 0 & 1\end{bmatrix}$.

24 $A = \begin{bmatrix}1 & 1\\ 0 & 3\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}\mathbf{1} & 0\\ 0 & \mathbf{3}\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix}$. Then $e^{At} = \begin{bmatrix}e^t & \frac{1}{2}(e^{3t} - e^t)\\ 0 & e^{3t}\end{bmatrix}$.

26 (a) The inverse of e^{At} is e^{-At} (b) If $Ax = \lambda x$ then $e^{At}x = e^{\lambda t}x$ and $e^{\lambda t} \neq 0$.

27 $(x, y) = (e^{4t}, e^{-4t})$ is a growing solution. The correct matrix for the exchanged $u = (y, x)$ is $\begin{bmatrix}2 & -2\\ -4 & 0\end{bmatrix}$. It *does* have the same eigenvalues as the original matrix.

28 Centering produces $U_{n+1} = \begin{bmatrix}1 & \Delta t\\ -\Delta t & 1 - (\Delta t)^2\end{bmatrix}U_n$. At $\Delta t = 1$, $\lambda = e^{i\pi/3}$ and $e^{-i\pi/3}$ both have $\lambda^6 = 1$ so $A^6 = I$. $U_6 = A^6U_0$ comes exactly back to U_0.

29 First A has $\lambda = \pm i$ and $A^4 = I$
Second A has $\lambda = -1, -1$ and $A^n = (-1)^n\begin{bmatrix}1 - 2n & -2n\\ 2n & 2n + 1\end{bmatrix}$ Linear growth.

30 With $a = \Delta t/2$ the trapezoidal step is $U_{n+1} = \dfrac{1}{1+a^2}\begin{bmatrix} 1-a^2 & 2a \\ -2a & 1-a^2 \end{bmatrix} U_n$.

Orthonormal columns $\Rightarrow$ orthogonal matrix $\Rightarrow \|U_{n+1}\| = \|U_n\|$

31 (a) $(\cos A)x = (\cos\lambda)x$ (b) $\lambda(A) = 2\pi$ and 0 so $\cos\lambda = 1, 1$ and $\cos A = I$ (c) $u(t) = 3(\cos 2\pi t)(1,1) + 1(\cos 0t)(1,-1)$ [$u' = Au$ has **exp**, $u'' = Au$ has **cos**]

Problem Set 6.4, page 337

3 $\lambda = 0, 4, -2$; unit vectors $\pm(0,1,-1)/\sqrt{2}$ and $\pm(2,1,1)/\sqrt{6}$ and $\pm(1,-1,-1)/\sqrt{3}$.

5 $Q = \dfrac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. The columns of Q are unit eigenvectors of A. Each unit eigenvector could be multiplied by -1

8 If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is symmetric then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ gives $\Lambda = 0$. The only symmetric A is $Q\,0\,Q^{\mathrm{T}}$ = zero matrix.

10 If x is not real then $\lambda = x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ is *not* always real. Can't assume real eigenvectors!

11 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2\begin{bmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{bmatrix} + 4\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0\begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25\begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

14 M is skew-symmetric and orthogonal; λ's must be $i, i, -i, -i$ to have trace zero.

16 (a) If $Az = \lambda y$ and $A^{\mathrm{T}}y = \lambda z$ then $B[y;\ -z] = [-Az;\ A^{\mathrm{T}}y] = -\lambda[y;\ -z]$. So $-\lambda$ is also an eigenvalue of B. (b) $A^{\mathrm{T}}Az = A^{\mathrm{T}}(\lambda y) = \lambda^2 z$. (c) $\lambda = -1, -1, 1, 1$; $x_1 = (1,0,-1,0)$, $x_2 = (0,1,0,-1)$, $x_3 = (1,0,1,0)$, $x_4 = (0,1,0,1)$.

19 A has $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B has $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$. Perpendicular for A. Not perpendicular for B since $B^{\mathrm{T}} \neq B$

21 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}}$ (c) True from $A^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

22 A and A^{T} have the same λ's but the *order* of the x's can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $x_1 = (1, i)$ first for A but $x_1 = (1, -i)$ first for A^{T}.

23 A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. A allows $QR, S\Lambda S^{-1}, Q\Lambda Q^{\mathrm{T}}$; B allows $S\Lambda S^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

24 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ if $b = 1$; repeated λ and no S if $b = -1$; singular if $b = 0$.

25 Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so $\lambda = \pm 1$. Then $A = \pm I$ or $A = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

27 The roots of $\lambda^2 + b\lambda + c = 0$ differ by $\sqrt{b^2 - 4c}$. For $\det(A + tB - \lambda I)$ we have $b = -3 - 8t$ and $c = 2 + 16t - t^2$. The minimum of $b^2 - 4c$ is $1/17$ at $t = 2/17$. Then $\lambda_2 - \lambda_1 = 1/\sqrt{17}$.

29 (a) $A = Q\Lambda\overline{Q}^{\mathrm{T}}$ times $\overline{A}^{\mathrm{T}} = Q\overline{\Lambda}^{\mathrm{T}}\overline{Q}^{\mathrm{T}}$ equals $\overline{A}^{\mathrm{T}}$ times A because $\Lambda\overline{\Lambda}^{\mathrm{T}} = \overline{\Lambda}^{\mathrm{T}}\Lambda$ (diagonal!) (b) step 2: The 1, 1 entries of $\overline{T}^{\mathrm{T}}T$ and $T\overline{T}^{\mathrm{T}}$ are $|a|^2$ and $|a|^2 + |b|^2$. This makes $b = 0$ and $T = \Lambda$.

30 a_{11} is $\left[q_{11} \dots q_{1n}\right]\left[\lambda_1\overline{q}_{11} \dots \lambda_n\overline{q}_{1n}\right]^{\mathrm{T}} \le \lambda_{\max}\left(|q_{11}|^2 + \dots + |q_{1n}|^2\right) = \lambda_{\max}$.

31 (a) $x^{\mathrm{T}}(Ax) = (Ax)^{\mathrm{T}}x = x^{\mathrm{T}}A^{\mathrm{T}}x = -x^{\mathrm{T}}Ax$. (b) $\overline{z}^{\mathrm{T}}Az$ is pure imaginary, its real part is $x^{\mathrm{T}}Ax + y^{\mathrm{T}}Ay = 0 + 0$ (c) $\det A = \lambda_1 \dots \lambda_n \ge 0$: pairs of λ's $= ib, -ib$.

Problem Set 6.5, page 350

3 Positive definite for $-3 < b < 3$ $\quad \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 9-b^2\end{bmatrix} = \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9-b^2\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}$

Positive definite for $c > 8$ $\quad \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\ 0 & c-8\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & c-8\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}$.

4 $f(x,y) = x^2 + 4xy + 9y^2 = (x+2y)^2 + 5y^2$; $x^2 + 6xy + 9y^2 = (x+3y)^2$.

8 $A = \begin{bmatrix}3 & 6\\ 6 & 16\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$. Pivots 3, 4 outside squares, ℓ_{ij} inside. $x^{\mathrm{T}}Ax = 3(x+2y)^2 + 4y^2$

10 $A = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $\quad B = \begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$ is singular; $B\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

12 A is positive definite for $c > 1$; determinants $c, c^2 - 1, (c-1)^2(c+2) > 0$. B is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

14 The eigenvalues of A^{-1} are positive because they are $1/\lambda(A)$. And the entries of A^{-1} pass the determinant tests. And $x^{\mathrm{T}}A^{-1}x = (A^{-1}x)^{\mathrm{T}}A(A^{-1}x) > 0$ for all $x \neq \mathbf{0}$.

17 If a_{jj} were smaller than all λ's, $A - a_{jj}I$ would have all eigenvalues > 0 (positive definite). But $A - a_{jj}I$ has a *zero* in the (j, j) position; impossible by Problem 16.

21 A is positive definite when $s > 8$; B is positive definite when $t > 5$ by determinants.

22 $R = \dfrac{\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}}{\sqrt{2}}\begin{bmatrix}\sqrt{9} & \\ & \sqrt{1}\end{bmatrix}\dfrac{\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}}{\sqrt{2}} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$; $R = Q\begin{bmatrix}4 & 0\\ 0 & 2\end{bmatrix}Q^{\mathrm{T}} = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$.

24 The ellipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $1/\sqrt{\lambda} = \sqrt{2}$ and $\sqrt{2/3}$.

25 $A = C^{\mathrm{T}}C = \begin{bmatrix}9 & 3\\ 3 & 5\end{bmatrix}$; $\begin{bmatrix}4 & 8\\ 8 & 25\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & 9\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ and $C = \begin{bmatrix}2 & 4\\ 0 & 3\end{bmatrix}$

29 $H_1 = \begin{bmatrix}6x^2 & 2x\\ 2x & 2\end{bmatrix}$ is positive definite if $x \neq 0$; $F_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$; $H_2 = \begin{bmatrix}6x & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ is indefinite, $(0, 1)$ is a saddle point of F_2.

31 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a "trough" staying at zero on the line $2x + 3y = 0$.

32 Orthogonal matrices, exponentials e^{At}, matrices with $\det = 1$ are groups. Examples of subgroups are orthogonal matrices with $\det = 1$, exponentials e^{An} for integer n.

34 The five eigenvalues of K are $2 - 2\cos\frac{k\pi}{6} = 2 - \sqrt{3}, 2 - 1, 2, 2 + 1, 2 + \sqrt{3}$: product of eigenvalues $= 6 = \det K$.

Problem Set 6.6, page 360

1 $B = GCG^{-1} = GF^{-1}AFG^{-1}$ so $M = FG^{-1}$. C similar to A and $B \Rightarrow A$ similar to B.

6 *Eight families* of similar matrices: six matrices have $\lambda = 0, 1$ (one family); three matrices have $\lambda = 1, 1$ and three have $\lambda = 0, 0$ (two families each!); one has $\lambda = 1, -1$; one has $\lambda = 2, 0$; two have $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$ (they are in one family).

7 (a) $(M^{-1}AM)(M^{-1}x) = M^{-1}(Ax) = M^{-1}0 = 0$ (b) The nullspaces of A and of $M^{-1}AM$ have the same *dimension*. Different vectors and different bases.

8 Same Λ Same S But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues $\lambda = 0, 0$.

10 $J^2 = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix}$ and $J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}$; $J^0 = I$ and $J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}$.

14 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1}J_iM_i = M_i^{\mathrm{T}}$ in each block (2) M_0 has those diagonal blocks M_i to get $M_0^{-1}JM_0 = J^{\mathrm{T}}$. (3) $A^{\mathrm{T}} = (M^{-1})^{\mathrm{T}}J^{\mathrm{T}}M^{\mathrm{T}}$ equals $(M^{-1})^{\mathrm{T}}M_0^{-1}JM_0M^{\mathrm{T}} = (MM_0M^{\mathrm{T}})^{-1}A(MM_0M^{\mathrm{T}})$, and A^{T} *is similar to* A.

17 (a) *False*: Diagonalize a nonsymmetric $A = S\Lambda S^{-1}$. Then Λ is symmetric and similar (b) *True*: A singular matrix has $\lambda = 0$. (c) *False*: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar (they have $\lambda = \pm 1$) (d) *True*: Adding I increases all eigenvalues by 1

18 $AB = B^{-1}(BA)B$ so AB is similar to BA. *If* $ABx = \lambda x$ *then* $BA(Bx) = \lambda(Bx)$.

19 Diagonal blocks 6 by 6, 4 by 4; AB has the same eigenvalues as BA plus $6 - 4$ zeros.

22 $A = MJM^{-1}, A^n = MJ^nM^{-1} = 0$ (each J^k has 1's on the kth diagonal). $\det(A - \lambda I) = \lambda^n$ so $J^n = 0$ by the Cayley-Hamilton Theorem.

Problem Set 6.7, page 371

1 $A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}} = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$

4 $A^{\mathrm{T}}A = AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \dfrac{3 + \sqrt{5}}{2}, \sigma_2^2 = \dfrac{3 - \sqrt{5}}{2}$. But A is indefinite $\sigma_1 = (1 + \sqrt{5})/2 = \lambda_1(A)$, $\sigma_2 = (\sqrt{5} - 1)/2 = -\lambda_2(A)$; $u_1 = v_1$ but $u_2 = -v_2$.

5 A proof that **eigshow** finds the SVD. When $V_1 = (1,0), V_2 = (0,1)$ the demo finds AV_1 and AV_2 at some angle θ. A 90° turn by the mouse to $V_2, -V_1$ finds AV_2 and $-AV_1$ at the angle $\pi - \theta$. Somewhere between, the constantly orthogonal v_1 and v_2 must produce Av_1 and Av_2 *at angle* $\pi/2$. Those orthogonal directions give u_1 and u_2.

9 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

14 The smallest change in A is to set its smallest singular value σ_2 to zero.

15 The singular values of $A + I$ are *not* $\sigma_j + 1$. Need eigenvalues of $(A + I)^{\mathrm{T}}(A + I)$.

17 $A = U\Sigma V^{\mathrm{T}}$ = [cosines including u_4] **diag(sqrt**$(2 - \sqrt{2}, 2, 2 + \sqrt{2})$**)** [sine matrix]$^{\mathrm{T}}$. $AV = U\Sigma$ says that differences of sines in V are cosines in U times σ's.

Problem Set 7.1, page 380

3 $T(v) = (0,1)$ and $T(v) = v_1 v_2$ are not linear.

4 (a) $S(T(v)) = v$ (b) $S(T(v_1) + T(v_2)) = S(T(v_1)) + S(T(v_2))$.

5 Choose $v = (1,1)$ and $w = (-1,0)$. $T(v) + T(w) = (0,1)$ but $T(v+w) = (0,0)$.

7 (a) $T(T(v)) = v$ (b) $T(T(v)) = v + (2,2)$ (c) $T(T(v)) = -v$ (d) $T(T(v)) = T(v)$.

10 Not invertible: (a) $T(1,0) = \mathbf{0}$ (b) $(0,0,1)$ is not in the range (c) $T(0,1) = \mathbf{0}$.

12 Write v as a combination $c(1,1) + d(2,0)$. Then $T(v) = c(2,2) + d(0,0)$. $T(v) = (4,4)$; $(2,2)$; $(2,2)$; if $v = (a,b) = b(1,1) + \frac{a-b}{2}(2,0)$ then $T(v) = b(2,2) + (0,0)$.

16 No matrix A gives $A\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To professors: Linear transformations on matrix space come from **4** by **4** matrices. Those in Problems 13–15 were special.

17 (a) True (b) True (c) True (d) False.

19 $T(T^{-1}(M)) = M$ so $T^{-1}(M) = A^{-1}MB^{-1}$.

20 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because $T(1,0) = (a_{11}, 0)$.

27 Also **30** emphasizes that circles are transformed to ellipses (see figure in Section 6.7).

29 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0,0)$.)

Problem Set 7.2, page 395

3 (Matrix $A)^2 = B$ when (transformation $T)^2 = S$ and output basis = input basis.

5 $T(v_1 + v_2 + v_3) = 2w_1 + w_2 + 2w_3$; A times $(1,1,1)$ gives $(2,1,2)$.

6 $v = c(v_2 - v_3)$ gives $T(v) = \mathbf{0}$; nullspace is $(0, c, -c)$; solutions $(1,0,0) + (0,c,-c)$.

8 For $T^2(v)$ we would need to know $T(w)$. If the w's equal the v's, the matrix is A^2.

12 (c) is wrong because w_1 is not generally in the input space.

14 (a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ = inverse of (a) (c) $A\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ must be $2A\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

16 $MN = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$.

18 $(a,b) = (\cos\theta, -\sin\theta)$. Minus sign from $Q^{-1} = Q^{\mathrm{T}}$.

20 $w_2(x) = 1 - x^2$; $w_3(x) = \frac{1}{2}(x^2 - x)$; $y = 4w_1 + 5w_2 + 6w_3$.

23 The matrix M with these nine entries must be invertible.

27 If T is not invertible, $T(v_1), \ldots, T(v_n)$ is not a basis. We couldn't choose $w_i = T(v_i)$.

30 S takes (x,y) to $(-x,y)$. $S(T(v)) = (\mathbf{-1}, \mathbf{2})$. $S(v) = (-2,1)$ and $T(S(v)) = (\mathbf{1}, \mathbf{-2})$.

34 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, −2, −2. Therefore $c_1 = 4$ and $c_2 = 2$ and $c_3 = 1$ and $c_4 = 1$.

35 The wavelet basis is $(1,1,1,1,1,1,1,1)$ and the long wavelet and two medium wavelets $(1,1,-1,-1,0,0,0,0)$, $(0,0,0,0,1,1,-1,-1)$ and 4 wavelets with a single pair $1,-1$.

36 If $V\boldsymbol{b} = W\boldsymbol{c}$ then $\boldsymbol{b} = V^{-1}W\boldsymbol{c}$. The change of basis matrix is $V^{-1}W$.

37 Multiplication by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with this basis is represented by 4 by 4 $A = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix}$

38 If $\boldsymbol{w}_1 = A\boldsymbol{v}_1$ and $\boldsymbol{w}_2 = A\boldsymbol{v}_2$ then $a_{11} = a_{22} = 1$. All other entries will be zero.

Problem Set 7.3, page 406

1 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has $\lambda = 50$ and 0, $\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\sigma_1 = \sqrt{50}$.

$A\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 5 \\ 15 \end{bmatrix} = \sigma_1\boldsymbol{u}_1$ and $A\boldsymbol{v}_2 = \mathbf{0}$. $\boldsymbol{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $AA^{\mathrm{T}}\boldsymbol{u}_1 = 50\,\boldsymbol{u}_1$.

3 $A = QH = \frac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix}\frac{1}{\sqrt{50}}\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. H is semidefinite because A is singular.

4 $A^+ = V\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}U^{\mathrm{T}} = \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; $A^+A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$, $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$.

7 $\begin{bmatrix} \sigma_1\boldsymbol{u}_1 & \sigma_2\boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \end{bmatrix} = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. In general this is $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r\boldsymbol{u}_r\boldsymbol{v}_r^{\mathrm{T}}$.

9 A^+ is A^{-1} because A is invertible. Pseudoinverse equals inverse when A^{-1} exists!

11 $A = [1]\,[5 \;\; 0 \;\; 0]V^{\mathrm{T}}$ and $A^+ = V\begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}$; $A^+A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $AA^+ = [1]$

13 If $\det A = 0$ then $\operatorname{rank}(A) < n$; thus $\operatorname{rank}(A^+) < n$ and $\det A^+ = 0$.

16 $\boldsymbol{x}^+$ in the row space of A is perpendicular to $\widehat{\boldsymbol{x}} - \boldsymbol{x}^+$ in the nullspace of $A^{\mathrm{T}}A =$ nullspace of A. The right triangle has $c^2 = a^2 + b^2$.

17 $AA^+\boldsymbol{p} = \boldsymbol{p}$, $AA^+\boldsymbol{e} = \mathbf{0}$, $A^+A\boldsymbol{x}_r = \boldsymbol{x}_r$, $A^+A\boldsymbol{x}_n = \mathbf{0}$.

19 L is determined by ℓ_{21}. Each eigenvector in S is determined by one number. The counts are $1+3$ for LU, $1+2+1$ for LDU, $1+3$ for QR, $1+2+1$ for $U\Sigma V^{\mathrm{T}}$, $2+2+0$ for $S\Lambda S^{-1}$.

22 Keep only the r by r corner Σ_r of Σ (the rest is all zero). Then $A = U\Sigma V^{\mathrm{T}}$ has the required form $A = \widehat{U}M_1\Sigma_r M_2^{\mathrm{T}}\widehat{V}^{\mathrm{T}}$ with an invertible $M = M_1\Sigma_r M_2^{\mathrm{T}}$ in the middle.

23 $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{v} \\ A^{\mathrm{T}}\boldsymbol{u} \end{bmatrix} = \sigma\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$. The singular values of A are *eigenvalues* of this block matrix.

Problem Set 8.1, page 418

3 The rows of the free-free matrix in equation (9) add to $[0\ \ 0\ \ 0]$ so the right side needs $f_1+f_2+f_3=0$. $f=(-1,0,1)$ gives $c_2u_1-c_2u_2=-1, c_3u_2-c_3u_3=-1, 0=0$. Then $u_{\text{particular}}=(-c_2^{-1}-c_3^{-1},-c_3^{-1},0)$. Add any multiple of $u_{\text{nullspace}}=(1,1,1)$.

4 $\displaystyle\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)dx=-\left[c(x)\frac{du}{dx}\right]_0^1=0$ (bdry cond) so we need $\displaystyle\int f(x)\,dx=0$.

6 Multiply $A_1^{\mathrm{T}}C_1A_1$ as columns of A_1^{T} times c's times rows of A_1. The first 3 by 3 "*element matrix*" $c_1E_1=[1\ \ 0\ \ 0]^{\mathrm{T}}c_1[1\ \ 0\ \ 0]$ has c_1 in the top left corner.

8 The solution to $-u''=1$ with $u(0)=u(1)=0$ is $u(x)=\frac{1}{2}(x-x^2)$. At $x=\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$ this gives $u=2,3,3,2$ (discrete solution in Problem 7) times $(\Delta x)^2=1/25$.

11 Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB: E = diag(ones(6, 1), 1); K = 64 ∗ (2∗ eye(7) − E − E'); D = 80 ∗ (E− eye(7)); (K + D)\ones(7, 1); % forward; (K − D')\ones(7, 1); % backward; (K + D/2 − D'/2)\ones(7, 1); % centered is usually the best: more accurate

Problem Set 8.2, page 428

1 $A=\begin{bmatrix}-1&1&0\\-1&0&1\\0&-1&1\end{bmatrix}$; nullspace contains $\begin{bmatrix}c\\c\\c\end{bmatrix}$; $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}y=\mathbf{0}$ for $y=(1,-1,1)$; current along edge 1, edge 3, back on edge 2 (full loop).

5 Kirchhoff's Current Law $A^{\mathrm{T}}y=f$ is solvable for $f=(1,-1,0)$ and not solvable for $f=(1,0,0)$; f must be orthogonal to $(1,1,1)$ in the nullspace: $f_1+f_2+f_3=0$.

6 $A^{\mathrm{T}}Ax=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}x=\begin{bmatrix}3\\-3\\0\end{bmatrix}=f$ produces $x=\begin{bmatrix}1\\-1\\0\end{bmatrix}+\begin{bmatrix}c\\c\\c\end{bmatrix}$; potentials $x=1,-1,0$ and currents $-Ax=2,1,-1$; f sends 3 units from node 2 into node 1.

7 $A^{\mathrm{T}}\begin{bmatrix}1&&\\&2&\\&&2\end{bmatrix}A=\begin{bmatrix}3&-1&-2\\-1&3&-2\\-2&-2&4\end{bmatrix}$; $f=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ yields $x=\begin{bmatrix}5/4\\1\\7/8\end{bmatrix}+$ any $\begin{bmatrix}c\\c\\c\end{bmatrix}$; potentials $x=\frac{5}{4},1,\frac{7}{8}$ and currents $-CAx=\frac{1}{4},\frac{3}{4},\frac{1}{4}$.

9 Elimination on $Ax=b$ always leads to $y^{\mathrm{T}}b=0$ in the zero rows of U and R: $-b_1+b_2-b_3=0$ and $b_3-b_4+b_5=0$ (those y's are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage* Law around the two *loops*.

11 $A^{\mathrm{T}}A=\begin{bmatrix}2&-1&-1&0\\-1&3&-1&-1\\-1&-1&3&-1\\0&-1&-1&2\end{bmatrix}$
diagonal entry = number of edges into the node
the trace is 2 times the number of nodes
off-diagonal entry = −1 if nodes are connected
$A^{\mathrm{T}}A$ is the **graph Laplacian**, $A^{\mathrm{T}}CA$ is **weighted** by C

13 $A^{\mathrm{T}}CAx=\begin{bmatrix}4&-2&-2&0\\-2&8&-3&-3\\-2&-3&8&-3\\0&-3&-3&6\end{bmatrix}x=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$
gives four potentials $x=(\frac{5}{12},\frac{1}{6},\frac{1}{6},0)$
I grounded $x_4=0$ and solved for x
currents $y=-CAx=(\frac{2}{3},\frac{2}{3},0,\frac{1}{2},\frac{1}{2})$

17 (a) 8 independent columns (b) f must be orthogonal to the nullspace so f's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

Problem Set 8.3, page 437

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; $A^\infty = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $x = (1, 0)$; 1 and $-.8$, $x = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

5 The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1)$ = everyone is dead.

6 Add the components of $Ax = \lambda x$ to find sum $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

7 $(.5)^k \to 0$ gives $A^k \to A^\infty$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $\begin{matrix} a \leq 1 \\ .4 + .6a \geq 0 \end{matrix}$

9 M^2 is still nonnegative; $[1 \ \cdots \ 1]M = [1 \ \cdots \ 1]$ so multiply on the right by M to find $[1 \ \cdots \ 1]M^2 = [1 \ \cdots \ 1] \Rightarrow$ columns of M^2 add to 1.

10 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $x_1 = (b, 1 - a)$.

12 B has $\lambda = 0$ and $-.5$ with $x_1 = (.3, \ .2)$ and $x_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t} x_1 = c_1 x_1$.

13 $x = (1, 1, 1)$ is an eigenvector when the row sums are equal; $Ax = (.9, .9, .9)$.

15 The first two A's have $\lambda_{\max} < 1$; $p = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

16 $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state $(.3, .3, .4)$ and $(30, 30, 40)$.

17 *No*, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

19 Λ times $S^{-1} \Delta S$ has the same diagonal as $S^{-1} \Delta S$ times Λ because Λ is diagonal.

20 If $B > A > 0$ and $Ax = \lambda_{\max}(A)x > 0$ then $Bx > \lambda_{\max}(A)x$ and $\lambda_{\max}(B) > \lambda_{\max}(A)$.

Problem Set 8.4, page 446

1 Feasible set = line segment $(6, 0)$ to $(0, 3)$; minimum cost at $(6, 0)$, maximum at $(0, 3)$.

2 Feasible set has corners $(0, 0), (6, 0), (2, 2), (0, 6)$. Minimum cost $2x - y$ at $(6, 0)$.

3 Only two corners $(4, 0, 0)$ and $(0, 2, 0)$; let $x_i \to -\infty$, $x_2 = 0$, and $x_3 = x_1 - 4$.

4 From $(0, 0, 2)$ move to $x = (0, 1, 1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $x = (1, 0, 1.5)$ with cost $5(1) + 8(1.5) = \$17$; $r = 1$ means more expensive.

5 $c = [3 \ \ 5 \ \ 7]$ has minimum cost 12 by the Ph.D. since $x = (4, 0, 0)$ is minimizing. The dual problem maximizes $4y$ subject to $y \leq 3$, $y \leq 5$, $y \leq 7$. Maximum = 12.

8 $y^{\mathrm{T}} b \leq y^{\mathrm{T}} A x = (A^{\mathrm{T}} y)^{\mathrm{T}} x \leq c^{\mathrm{T}} x$. The first inequality needed $y \geq 0$ and $Ax - b \geq 0$.

Problem Set 8.5, page 451

1 $\int_0^{2\pi} \cos((j+k)x)\,dx = \left[\frac{\sin((j+k)x)}{j+k}\right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos((j-k)x)\,dx = 0$ Notice $j-k \neq 0$ in the denominator. If $j=k$ then $\int_0^{2\pi} \cos^2 jx\,dx = \pi$.

4 $\int_{-1}^{1}(1)(x^3-cx)\,dx = 0$ and $\int_{-1}^{1}(x^2-\frac{1}{3})(x^3-cx)\,dx = 0$ for all c (odd functions). Choose c so that $\int_{-1}^{1} x(x^3-cx)\,dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^{1} = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.

5 The integrals lead to the Fourier coefficients $a_1 = 0,\ b_1 = 4/\pi,\ b_2 = 0$.

6 From eqn. (3) $a_k = 0$ and $b_k = 4/\pi k$ *(odd k)*. The square wave has $\|f\|^2 = 2\pi$. Then eqn. (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots)$. That infinite series equals $\pi^2/8$.

8 $\|\boldsymbol{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ so $\|\boldsymbol{v}\| = \sqrt{2}$; $\|\boldsymbol{v}\|^2 = 1 + a^2 + a^4 + \cdots = 1/(1-a^2)$ so $\|\boldsymbol{v}\| = 1/\sqrt{1-a^2}$; $\int_0^{2\pi}(1 + 2\sin x + \sin^2 x)\,dx = 2\pi + 0 + \pi$ so $\|f\| = \sqrt{3\pi}$.

9 (a) $f(x) = (1 + \text{square wave})/2$ so the a's are $\frac{1}{2}$, 0, 0, … and the b's are $2/\pi$, 0, $-2/3\pi, 0, 2/5\pi, \ldots$ (b) $a_0 = \int_0^{2\pi} x\,dx/2\pi = \pi$, all other $a_k = 0, b_k = -2/k$.

11 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$; $\cos(x + \frac{\pi}{3}) = \cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$.

13 $a_0 = \dfrac{1}{2\pi}\int F(x)\,dx = \dfrac{1}{2\pi}, a_k = \dfrac{\sin(kh/2)}{\pi kh/2} \to \dfrac{1}{\pi}$ for delta function; all $b_k = 0$.

Problem Set 8.6, page 458

3 If $\sigma_3 = 0$ the third equation is exact.

4 0, 1, 2 have probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ and $\sigma^2 = (0-1)^2\frac{1}{4} + (1-1)^2\frac{1}{2} + (2-1)^2\frac{1}{4} = \frac{1}{2}$.

5 Mean $(\frac{1}{2}, \frac{1}{2})$. Independent flips lead to $\boldsymbol{\Sigma} = \textbf{diag}(\frac{1}{4}, \frac{1}{4})$. Trace $= \sigma^2_{\text{total}} = \frac{1}{2}$.

6 Mean $m = p_0$ and variance $\sigma^2 = (1-p_0)^2 p_0 + (0-p_0)^2(1-p_0) = p_0(1-p_0)$.

7 Minimize $P = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$ at $P' = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0; a = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$ recovers equation (2) for the statistically correct choice with minimum variance.

8 Multiply $L\Sigma L^{\mathrm{T}} = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}A^{\mathrm{T}}\Sigma^{-1}\Sigma\Sigma^{-1}A(A^{\mathrm{T}}\Sigma^{-1}A)^{-1} = P = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}$.

9 Row 3 = −row 1 and row 4 = −row 2: A has rank 2.

Problem Set 8.7, page 464

1 (x, y, z) has homogeneous coordinates (cx, cy, cz, c) for $c = 1$ and all $c \neq 0$.

4 $S = \text{diag}(c, c, c, 1)$; row 4 of ST and TS is $1, 4, 3, 1$ and $c, 4c, 3c, 1$; use $\boldsymbol{v}TS$!

5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square, starting from an 8.5 by 11 page.

9 $\boldsymbol{n} = \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3}\right)$ has $P = I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \dfrac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$. Notice $\|\boldsymbol{n}\| = 1$.

10 We can choose $(0,0,3)$ on the plane and multiply $T_-PT_+ = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}$.

11 $(3,3,3)$ projects to $\frac{1}{3}(-1,-1,4)$ and $(3,3,3,1)$ projects to $(\frac{1}{3},\frac{1}{3},\frac{5}{3},1)$. Row vectors!

13 That projection of a cube onto a plane produces a hexagon.

14 $(3,3,3)(I-2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = \left(-\frac{11}{3},-\frac{11}{3},-\frac{1}{3}\right)$.

15 $(3,3,3,1) \to (3,3,0,1) \to (-\frac{7}{3},-\frac{7}{3},-\frac{8}{3},1) \to (-\frac{7}{3},-\frac{7}{3},\frac{1}{3},1)$.

17 Space is rescaled by $1/c$ because (x,y,z,c) is the same point as $(x/c,y/c,z/c,1)$.

Problem Set 9.1, page 472

1 Without exchange, pivots .001 and 1000; with exchange, 1 and -1. When the pivot is larger than the entries below it, all $|\ell_{ij}| = |\text{entry/pivot}| \le 1$. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

4 The largest $\|\boldsymbol{x}\| = \|A^{-1}\boldsymbol{b}\|$ is $\|A^{-1}\| = 1/\lambda_{\min}$ since $A^{\mathrm{T}} = A$; largest error $10^{-16}/\lambda_{\min}$.

5 Each row of U has at most w entries. Then w multiplications to substitute components of $\boldsymbol{x}$ (already known from below) and divide by the pivot. Total for n rows $< wn$.

6 The triangular L^{-1}, U^{-1}, R^{-1} need $\frac{1}{2}n^2$ multiplications. Q needs n^2 to multiply the right side by $Q^{-1} = Q^{\mathrm{T}}$. So $QR\boldsymbol{x} = \boldsymbol{b}$ takes 1.5 times longer than $LU\boldsymbol{x} = \boldsymbol{b}$.

7 $UU^{-1} = I$: Back substitution needs $\frac{1}{2}j^2$ multiplications on column j, using the j by j upper left block. Then $\frac{1}{2}(1^2+2^2+\cdots+n^2) \approx \frac{1}{2}(\frac{1}{3}n^3) =$ total to find U^{-1}.

10 With 16-digit floating point arithmetic the errors $\|\boldsymbol{x}-\boldsymbol{x}_{\text{computed}}\|$ for $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$ are of order $10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}$.

11 (a) $\cos\theta = \frac{1}{\sqrt{10}}$, $\sin\theta = \frac{-3}{\sqrt{10}}$, $R = Q_{21}A = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$ (b) $\lambda = 4$; use $-\theta$ $\boldsymbol{x} = (1,-3)/\sqrt{10}$

13 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos\theta$, the entries 1 and $\pm\tan\theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR.

Problem Set 9.2, page 478

1 $\|A\| = 2$, $\|A^{-1}\| = 2, c = 4$; $\|A\| = 3$, $\|A^{-1}\| = 1, c = 3$; $\|A\| = 2+\sqrt{2} = \lambda_{\max}$ for positive definite A, $\|A^{-1}\| = 1/\lambda_{\min}$, $c = (2+\sqrt{2})/(2-\sqrt{2}) = 5.83$.

3 For the first inequality replace $\boldsymbol{x}$ by $B\boldsymbol{x}$ in $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$; the second inequality is just $\|B\boldsymbol{x}\| \le \|B\|\|\boldsymbol{x}\|$. Then $\|AB\| = \max(\|AB\boldsymbol{x}\|/\|\boldsymbol{x}\|) \le \|A\|\|B\|$.

7 The triangle inequality gives $\|A\boldsymbol{x}+B\boldsymbol{x}\| \le \|A\boldsymbol{x}\| + \|B\boldsymbol{x}\|$. Divide by $\|\boldsymbol{x}\|$ and take the maximum over all nonzero vectors to find $\|A+B\| \le \|A\| + \|B\|$.

8 If $Ax = \lambda x$ then $\|Ax\|/\|x\| = |\lambda|$ for that particular vector x. When we maximize the ratio over all vectors we get $\|A\| \geq |\lambda|$.

13 The residual $b - Ay = (10^{-7}, 0)$ is much smaller than $b - Az = (.0013, .0016)$. But z is much closer to the solution than y.

14 $\det A = 10^{-6}$ so $A^{-1} = 10^3 \begin{bmatrix} 659 & -563 \\ -913 & 780 \end{bmatrix}$: $\|A\| > 1$, $\|A^{-1}\| > 10^6$, then $c > 10^6$.

16 $x_1^2 + \cdots + x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $(|x_1| + \cdots + |x_n|)^2 = \|x\|_1^2$. $x_1^2 + \cdots + x_n^2 \leq n \max(x_i^2)$ so $\|x\| \leq \sqrt{n}\|x\|_\infty$. Choose $y_i = \operatorname{sign} x_i = \pm 1$ to get $\|x\|_1 = x \cdot y \leq \|x\|\|y\| = \sqrt{n}\|x\|$. $x = (1, \ldots, 1)$ has $\|x\|_1 = \sqrt{n}\,\|x\|$.

Problem Set 9.3, page 489

2 If $Ax = \lambda x$ then $(I - A)x = (1 - \lambda)x$. Real eigenvalues of $B = I - A$ have $|1 - \lambda| < 1$ provided λ is between 0 and 2.

6 Jacobi has $S^{-1}T = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{3}$. Small problem, fast convergence.

7 Gauss-Seidel has $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{9}$ which is $(|\lambda|_{\max}$ for Jacobi$)^2$.

9 Set the trace $2 - 2\omega + \frac{1}{4}\omega^2$ equal to $(\omega - 1) + (\omega - 1)$ to find $\omega_{\text{opt}} = 4(2 - \sqrt{3}) \approx 1.07$. The eigenvalues $\omega - 1$ are about .07, a big improvement.

15 In the jth component of Ax_1, $\lambda_1 \sin \frac{j\pi}{n+1} = 2 \sin \frac{j\pi}{n+1} - \sin \frac{(j-1)\pi}{n+1} - \sin \frac{(j+1)\pi}{n+1}$. The last two terms combine into $-2 \sin \frac{j\pi}{n+1} \cos \frac{\pi}{n+1}$. Then $\lambda_1 = 2 - 2 \cos \frac{\pi}{n+1}$.

17 $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ gives $u_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $u_2 = \frac{1}{9}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $u_3 = \frac{1}{27}\begin{bmatrix} 14 \\ 13 \end{bmatrix} \to u_\infty = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

18 $R = Q^{\mathrm{T}}A = \begin{bmatrix} 1 & \cos\theta \sin\theta \\ 0 & -\sin^2\theta \end{bmatrix}$ and $A_1 = RQ = \begin{bmatrix} \cos\theta(1 + \sin^2\theta) & -\sin^3\theta \\ -\sin^3\theta & -\cos\theta \sin^2\theta \end{bmatrix}$.

20 If $A - cI = QR$ then $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$. No change in eigenvalues because A_1 is similar to A.

21 Multiply $Aq_j = b_{j-1}q_{j-1} + a_j q_j + b_j q_{j+1}$ by q_j^{T} to find $q_j^{\mathrm{T}}Aq_j = a_j$ (because the q's are orthonormal). The matrix form (multiplying by columns) is $AQ = QT$ where T is *tridiagonal*. The entries down the diagonals of T are the a's and b's.

23 If A is symmetric then $A_1 = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is also symmetric. $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$ has R and R^{-1} upper triangular, so A_1 cannot have nonzeros on a lower diagonal than A. If A is tridiagonal and symmetric then (by using symmetry for the upper part of A_1) the matrix $A_1 = RAR^{-1}$ is also tridiagonal.

26 If each center a_{ii} is larger than the circle radius r_i (this is diagonal dominance), then 0 is outside all circles: not an eigenvalue so A^{-1} exists.

Problem Set 10.1, page 498

2 In polar form these are $\sqrt{5}e^{i\theta}$, $5e^{2i\theta}$, $\frac{1}{\sqrt{5}}e^{-i\theta}$, $\sqrt{5}$.

4 $|z \times w| = 6$, $|z + w| \le 5$, $|z/w| = \frac{2}{3}$, $|z - w| \le 5$.

5 $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$; $w^{12} = 1$.

9 $2+i$; $(2+i)(1+i) = 1+3i$; $e^{-i\pi/2} = -i$; $e^{-i\pi} = -1$; $\frac{1-i}{1+i} = -i$; $(-i)^{103} = i$.

10 $z + \overline{z}$ is real; $z - \overline{z}$ is pure imaginary; $z\overline{z}$ is positive; $z/\overline{z}$ has absolute value 1.

12 (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the λ's can be real and different.

13 Complex λ's when $(a+d)^2 < 4(ad-bc)$; write $(a+d)^2-4(ad-bc)$ as $(a-d)^2+4bc$ which is positive when $bc > 0$.

14 $\det(P - \lambda I) = \lambda^4 - 1 = 0$ has $\lambda = 1, -1, i, -i$ with eigenvectors $(1, 1, 1, 1)$ and $(1, -1, 1, -1)$ and $(1, i, -1, -i)$ and $(1, -i, -1, i)$ = columns of Fourier matrix.

16 The symmetric block matrix has real eigenvalues; so $i\lambda$ is real and λ is pure imaginary.

18 $r = 1$, angle $\frac{\pi}{2} - \theta$; multiply by $e^{i\theta}$ to get $e^{i\pi/2} = i$.

21 $\cos 3\theta = \text{Re}[(\cos\theta + i\sin\theta)^3] = \cos^3\theta - 3\cos\theta\sin^2\theta$; $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

23 e^i is at angle $\theta = 1$ on the unit circle; $|i^e| = 1^e$; Infinitely many $i^e = e^{i(\pi/2+2\pi n)e}$.

24 (a) Unit circle (b) Spiral in to $e^{-2\pi}$ (c) Circle continuing around to angle $\theta = 2\pi^2$.

Problem Set 10.2, page 506

3 z = multiple of $(1+i, 1+i, -2)$; $Az = \mathbf{0}$ gives $z^H A^H = \mathbf{0}^H$ so z (not $\overline{z}$!) is orthogonal to all columns of A^H (using complex inner product z^H times columns of A^H).

4 The four fundamental subspaces are now $\boldsymbol{C}(A), \boldsymbol{N}(A), \boldsymbol{C}(A^H), \boldsymbol{N}(A^H)$. A^H ***and not*** A^T.

5 (a) $(A^H A)^H = A^H A^{HH} = A^H A$ again (b) If $A^H A z = \mathbf{0}$ then $(z^H A^H)(Az) = 0$. This is $\|Az\|^2 = 0$ so $Az = \mathbf{0}$. The nullspaces of A and $A^H A$ are always the ***same***.

6 (a) False (c) False $A = U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue when $A = A^H$.

10 $(1, 1, 1)$, $(1, e^{2\pi i/3}, e^{4\pi i/3})$, $(1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because P is an orthogonal matrix—and therefore its eigenvector matrix is unitary.

11 $C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} = 2 + 5P + 4P^2$ has the Fourier eigenvector matrix F.

The eigenvalues are $2 + 5 + 4 = 11$, $2 + 5e^{2\pi i/3} + 4e^{4\pi i/3}$, $2 + 5e^{4\pi i/3} + 4e^{8\pi i/3}$.

13 Determinant = product of the eigenvalues *(all real)*. And $A = A^H$ gives $\det A = \overline{\det A}$.

15 $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$.

18 $V = \frac{1}{L}\begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{L}\begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6+2\sqrt{3}$.
Unitary means $|\lambda| = 1$. $V = V^{\mathrm{H}}$ gives real λ. Then trace zero gives $\lambda = 1$ and -1.

19 The $\boldsymbol{v}$'s are columns of a unitary matrix U, so U^{H} is U^{-1}. Then $\boldsymbol{z} = UU^{\mathrm{H}}\boldsymbol{z} =$ (multiply by columns) $= \boldsymbol{v}_1(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z}) + \cdots + \boldsymbol{v}_n(\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})$: a typical orthonormal expansion.

20 Don't multiply $(e^{-ix})(e^{ix})$. Conjugate the first, then $\int_0^{2\pi} e^{2ix}\,dx = [e^{2ix}/2i]_0^{2\pi} = 0$.

21 $R + iS = (R + iS)^{\mathrm{H}} = R^{\mathrm{T}} - iS^{\mathrm{T}}$; R is symmetric but S is skew-symmetric.

24 $[\,1\,]$ and $[-1\,]$; any $[e^{i\theta}]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\overline{z} \\ -z & e^{i\phi}\overline{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$ and any angle ϕ

27 Unitary $U^{\mathrm{H}}U = I$ means $(A^{\mathrm{T}} - iB^{\mathrm{T}})(A + iB) = (A^{\mathrm{T}}A + B^{\mathrm{T}}B) + i(A^{\mathrm{T}}B - B^{\mathrm{T}}A) = I$. $A^{\mathrm{T}}A + B^{\mathrm{T}}B = I$ and $A^{\mathrm{T}}B - B^{\mathrm{T}}A = 0$ which makes the block matrix orthogonal.

30 $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S\Lambda S^{-1}$. Note real $\lambda = 1$ and 4.

Problem Set 10.3, page 514

8 $\boldsymbol{c} \to (1,1,1,1,0,0,0,0) \to (4,0,0,0,0,0,0,0) \to (4,0,0,0,4,0,0,0) = F_8\boldsymbol{c}$.
$\boldsymbol{C} \to (0,0,0,0,1,1,1,1) \to (0,0,0,0,4,0,0,0) \to (4,0,0,0,-4,0,0,0) = F_8\boldsymbol{C}$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1: Key to FFT.

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$; E contains the four eigenvalues of $C = FEF^{-1}$ because F contains the eigenvectors.

14 Eigenvalues $e_1 = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2$. Just transform column 0 of C. Check trace $0 + 2 + 4 + 2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2}n\log_2 n$ multiplications each by the **FFT**. The total is much less than the ordinary n^2 for C times $\boldsymbol{x}$.